Technical Appendix to Accompany 'On Risk Aversion and Investment – a Theoretical Approach' by John Fender and Peter Sinclair

(Not intended for publication, but to be made available to interested readers on request.)

A. **The Absence of a Credible Good-State Default Threat**

We show here that there cannot be a credible good-state default threat. (Strictly, lenders will not offer borrowers a level of debt that would give them a credible good-state default threat.) First of all, for default *not* to be credibly threatened in the good state, then the reverse of (8) with p_G replacing p_i must hold:

$$p_G F(K) > L(1+r). \tag{A1}$$

Using (12) changes (A1) to

$$\pi p_G F(K) > L - (1-\pi)(1-\alpha) p_B F(K). \tag{A2}$$

Reversing this inequality, the condition for default to be threatened credibly in the good state becomes (using the definition $p \equiv \pi p_G + (1-\pi)p_B$):

$$\left[p - \alpha(1-\pi) p_B\right] F(K) \leq L. \tag{A3}$$

Assume that were bargaining to take place in the good state, repayments would be determined as in (9) with p_G replacing p_B. Then, for lenders to expect to break even with bargaining in both states, the following inequality must be satisfied:

$$(1-\alpha) pF(K) \geq L. \tag{A4}$$

For positive α, π and p_G, $(1-\alpha)p = p - \alpha\{\pi p_G + (1-\pi)p_B\} < p - \alpha(1-\pi)p_B$, so (A3) and (A4) are incompatible with each other. This means that if the loan is such that the borrower has a credible default threat in both states, the lender will not get a competitive return on his loan from the renegotiation in the two states so will not extend the loan in the first place.

When α is zero, the borrower would get nothing in either state and the ICC would not be satisfied.

B. **The Outcome of Renegotiation**

Assume a Nash bargain determines the outcome. The lender receives R from the renegotiation, the borrower $p_B F(K) - R$. Let the borrower's utility function be $U(x) = x^y$, and the lender's (proportional to) R so the threat point of each party is zero. Then the repayment is determined by the value of R that maximises

$$R^{1-z}\left[p_B F(K) - R\right]^{yz} \tag{B1}$$

The bargaining strength of the borrower (lender) is given by z (1 – z). The first-order condition for maximising (B1) with respect to R is

$$(1-z)R^{-z}\left[p_B F(K) - R\right]^{yz} = R^{1-z} yz\left[p_B F(K) - R\right]^{yz-1}. \tag{B2}$$

This simplifies to

$$(1-z)\left[p_B F(K) - R\right] = yzR. \tag{B3}$$

So
$$R = \frac{1-z}{1-z+yz} p_B F(K). \tag{B4}$$

If we define $\alpha \equiv yz/(1-z+yz)$, we obtain (9).

C. The Game between the Entrepreneur and Lenders: the Criterion for Default to be Threatened Credibly.

We describe the game between entrepreneur and lenders in the case where the entrepreneur incurs debt. The game where lenders make an initial payment to the entrepreneur as well as financing the investment in exchange for an equity share is straightforward, as is the analysis of the choice between the two options.

1. An entrepreneur (T) meets a potential lender (N). An offer may be made (by N) whereby T commits to an investment of K, for which N provides a loan of L, with interest rate r. If no

offer is made, T either withdraws or approaches another lender. If an offer is made, T either accepts it, or turns it down and either approaches another lender or withdraws. This continues until an offer is accepted or T withdraws completely. If a loan offer is accepted and L is strictly less than K, T raises the remainder of the finance for the project from equity holders (S) in exchange for a stake e in the project at price Q, where $Q = pF(K) - L$ as in the paper (that is, equity is priced efficiently); this entitles S to a fraction e of the net receipts from the project; e is chosen to satisfy the financing constraint, (14).

2. After the investment has been made, T decides whether to contribute a certain amount of labour time to the project, at cost to him of Ψ. If he decides not to, the project terminates, and nobody receives anything. If he does, then the project is initiated and T enters the next period with capital stock K.

3. At the beginning of the second period the price is revealed to be p_G or p_B.

4. N requests repayment of the loan.

5. T *either* accepts the request, in which case the project is implemented, yielding total revenue $p_iF(K)$; T pays $min[L(1 + r), p_iF(K)]$ to N and $max[e\{p_iF(K) - L(1 + r)\}, 0]$ to S, consumes the residual and the game ends, *or* turns down the request, in which case we move to stage 6.

6. N *either* proposes renegotiation (go to stage 7) *or* continues to demand repayment of the full amount due (go to stage 8).

7. If N has proposed renegotiation at stage 6, T *either* accepts the proposal, renegotiation occurs, the parties agree on a payment of R to N and the project is implemented; total receipts $p_iF(K)$ are generated, of which N receives R, S receives $e[p_iF(K) - R]$ and T receives $(1 - e)[p_iF(K) - R]$ and the game ends, *or* rejects the proposal (go to stage 8). R is given by equation (9); Section B of the Technical Appendix above shows how it can be derived from more basic assumptions.

8. If renegotiation does not take place, either because N continues to demand repayment of the full amount due to him, or because T rejects the proposal of renegotiation, T decides whether to implement the project.

9. If the project *is* implemented (without renegotiation), then $p_iF(K)$ is produced, of which N obtains $min[L(1 + r), p_iF(K)]$, leaving T with $max[\{p_iF(K) - L(1 + r)\}(1 - e), 0]$. If the project *is not* implemented, N, S and T receive nothing.

T decides to implement the project at stage 9 iff $p_iF(K) - L(1 + r) > 0$. If the project is implemented in this case, then the loan is repaid in full, and T receives more (the LHS of the expression) than he would have had he not implemented the project (the RHS, zero). However, if $p_iF(K) \leq L(1 + r)$, T receives nothing regardless of whether it is implemented or not. We assume (see footnote 6 of the paper for a discussion of this assumption) that in this case the entrepreneur does not implement the project.

Will N propose renegotiation at stage 6? If $p_iF(K) > L(1 + r)$, then he knows that T will implement the project at stage 7, and N will receive more than he would under renegotiation; he does not therefore propose renegotiation if this condition is satisfied.

If $p_iF(K) \leq L(1 + r)$, and renegotiation *does not* take place, T will not implement the project, and T and N receive nothing. On the other hand, if $p_iF(K) \leq L(1 + r)$ and renegotiation *does* take place, then T receives $(1 - e)(p_iF(K) - R)$ and N receives R (both of which are positive). So T will accept N's proposal of renegotiation. Also, N will propose renegotiation, since his proposal will be accepted and his return from doing so, R, is greater than the return from not doing so, which is zero. If N continues to demand repayment of the full amount the project will not be implemented, and he will receive nothing.

What happens at stage 5? T will only reject the request of repayment if he can do better by so doing, that is if $p_iF(K) \leq L(1 + r)$.

At stage 2, the entrepreneur will implement the project if and only if the expected return to him from proceeding is at least Ψ.

The game hence has a subgame perfect equilibrium, the exact nature of which depends on whether the return on the project, less the loan repayment, is greater or less than 0. If $p_iF(K) > L(1 + r)$, the project is implemented and the loan repaid in full. If $p_iF(K) \leq L(1 + r)$, then renegotiation takes place, the project is implemented and the renegotiated payment is made.

D. Proof that in the Absence of Bad-State Renegotiation, Pure-Equity Finance will be Chosen.

If there is no bad-state renegotiation (which, of course, means no good-state renegotiation), then good- and bad-state consumption will be given by

$$C_G = (1-e)\left[p_G F(K) - L\right] \tag{D1}$$

$$C_B = (1-e)\left[p_B F(K) - L\right] \tag{D2}$$

Since there is no default, our assumptions on intermediation imply that the interest rate on loans is the safe rate, zero. *K*, *L* and *e* are related by equation (14):

$$K = (1-e)L + epF(K) \tag{D3}$$

Substituting from (D3) into (D1) and (D2), we obtain:

$$C_G = \left[(1-e)p_G + ep\right]F(K) - K \tag{D4}$$

$$C_B = \left[(1-e)p_B + ep\right]F(K) - K \tag{D5}$$

Substituting (D4) and (D5) into the expression for expected utility, $\pi U(C_G) + (1-\pi)U(C_B)$ - Ψ, and differentiating with respect to *e*, we obtain the following expression for the effect of an increase in *e* on expected utility:

$$\partial U / \partial e = \pi U'(C_G)(p - p_G) + (1-\pi)U'(C_B)(p - p_B) \tag{D6}$$

Using the definition of *p*, this becomes

$$\partial U / \partial e = \pi(1-\pi)(p_B - p_G)\left[U'(C_G) - U'(C_B)\right] \tag{D7}$$

Since $p_G > p_B$ and, provided $e < 1$, $U'(C_G) < U'(C_B)$, (D7) is always positive. So increasing the equity share will always raise expected utility. So if there is no bad-state renegotiation, the entrepreneur always has an incentive to raise *e*, given *K*, which, from (D3) means a fall in *L*. So the entrepreneur will raise *e* at the expense of *L* until *L* is reduced to zero (and hence pure-equity finance is used).

E. When will Each Type of Equilibrium Obtain?

To facilitate the discussion it is helpful if we refer to each type of equilibrium as a 'regime', designated by a particular term, as follows:

R1. The-pure equity solution.

R2. The first-best solution with debt and equity.

R3. Default condition just binding with $e > 0$.

R4. Default condition just binding with $e = 0$.

R5. Underinsurance with $e = 0$.

We here investigate the conditions under which each regime arises. Each regime will exist for certain parameter values. First, we examine which one of the renegotiation regimes (2, 3, 4 or 5) obtains, should renegotiation occur. We then examine the choice between the relevant renegotiation regime and R1, where there is no renegotiation. As before, we assume that the ICC is satisfied for the renegotiation regimes. First of all, when will the first-best solution (R2) obtain? This is characterised by levels of K, L and e such that (i) investment is efficient; (ii) consumption is equalised across states; (iii) the default condition is satisfied; and (iv) the equity share (e) is nonnegative.

The condition that consumption in the two states is equal is equation (18), repeated here for convenience:

$$L = \left[p - \alpha p_B\right] F\left(K\right). \tag{E1}$$

So, if (E1) is to hold for nonnegative e (that is, for $L \leq K$) when investment is efficient:

$$K^* \geq \left\{p - \alpha p_B\right\} F\left(K^*\right). \tag{E2}$$

This is our first condition for R2, combining conditions (i), (ii) and (iv). The second condition combines (i), (ii) and (iii). To obtain this, first combine the default condition (8) with i = B with (12) and manipulate to give

$$L \geq p_B F(K)\left[1 - \alpha(1 - \pi)\right] \tag{E3}$$

Combining (E1) and (E3) gives a condition for the default threat to be credible in the bad state when renegotiation in this state equalises consumption between the states:

$$p_G \geq \left(1+\alpha\right) p_B. \tag{E4}$$

So R2 obtains if (E2) and (E4) are both satisfied (that is, the default threat is credible and the equity share is nonnegative when the debt-equity ratio is such as to equalise consumption between the two states, assuming bad-state renegotiation).

Suppose, now, that condition (E4) fails. This means the default threat is not credible for the debt that equalises consumption between the states under bad-state renegotiation. If condition (E2) holds, then we have Regime 3 with efficient investment; if not, then we have overinvestment and Regime 4.

What happens if, instead, (E4) is satisfied but (E2) is not? Here consumption cannot be equalised between states as this would violate the constraint that equity be nonnegative. Instead, the firm has an incentive to borrow more, using the finance to increase the capital stock above its efficient level until the first-order condition (B8) in the Appendix of the paper is satisfied. R5 obtains.

The above conditions determine which of the renegotiation regimes obtains, should renegotiation occur. To determine whether there will in fact be renegotiation, we need to compare the level of expected utility in R1 with that obtained in the relevant renegotiation regime. However, since R1 never generates the first-best level of expected utility (for positive Ψ) and R2 does, if conditions (E2) and (E4) are both satisfied, R2, rather than R1, must obtain. However, if the conditions for R3, R4 or R5 are satisfied, then it is not clear which dominates; it depends on the value of Ψ - for small values of Ψ, R1 is more likely, but for larger values of Ψ, we would expect to see R3, R4 or R5.

F. An Example.

Let α = ¾, π= 1/2 and suppose the production function takes the form: $F(K) = K^{\beta}$. Then the efficient level of capital (that maximises 'surplus', $pF(K) – K$) is given by: $K^* = \{p\beta\}^{1/(1-\beta)}$. (E2) hence becomes

$$\frac{p}{p-\alpha p_B} \geq \frac{1}{\beta}. \tag{F1}$$

(E4) is unaltered, but for convenience we repeat it below:

$$p_G \geq (1+\alpha) p_B. \tag{F2}$$

We first consider which of the renegotiation regimes exist, should renegotiation take place; we then consider the choice between the relevant renegotiation regime and R1.

R2 will obtain if both (F1) and (F2) are satisfied. It can easily be checked that if $p_G = 2p_B$, then both conditions are satisfied provided that $\beta \geq$ ½.

R3 will obtain if (F1) is satisfied but (F2) is violated. This will be the case if $p_G = (3/2)p_B$ and $\beta \geq 0.4$.

R4 will obtain if neither (F1) nor (F2) is satisfied. This will be the case if $\beta < 0.4$ and $p_G = (3/2)p_B$.

R5 will obtain if (F1) is not satisfied but (F2) is satisfied. This will be the case for $p_G = 2p_B$ and $\beta <$ ½.

Suppose have β = ½, $p_G = 2p_B$ and $p = 2$, so $p_G = 8/3$, $p_B = 4/3$ and $K^* = F(K^*) = 1$. Then, from (E1), the loan that equalises consumption between states is 1 and the optimal equity share zero. We are on the borderline between R2 and R5. Good-state consumption is given by (16) and is 1, and bad-state consumption can be shown to also be 1 from (17). We hence have efficient investment and perfect insurance, the first best. Let the utility function be $U(x) = 2x^{1/2}$. Then expected utility is 2 - Ψ; this is also the first best.

We now need to consider what happens in the non-renegotiation regime, R1. From (4), with efficient investment, the relationship between the initial payment and the equity share is given by

$$Z = 2e - 1. \tag{F3}$$

Good-state consumption is hence (from (2)) $(5 - 2e)/3$ and similarly bad-state consumption is $(1 + 2e)/3$. It is clear that an equity share of 1, if possible, would equalise consumption between states; also, good-state consumption is decreasing in the equity share while bad-state consumption is increase in the equity share; these results are entirely as expected. Expected utility (from (6)) is hence:

$$\left\{(5-2e)/3\right\}^{1/2} + \left\{(1+2e)/3\right\}^{1/2} - \Psi \tag{F4}$$

and the ICC (equation (7)) can be written:

$$\left\{(5-2e)/3\right\}^{1/2} + \left\{(1+2e)/3\right\}^{1/2} - 2\left\{2e-1\right\}^{1/2} = \Psi \tag{F5}$$

This defines a relationship between e and Ψ, showing, for any given value of Ψ the maximum equity share that is compatible with the ICC. If e = ¾, then Ψ would be (approximately) 0.074. In this case, expected utility would be $(3.5/3)^{1/2} + (2.5/3)^{1/2} - \Psi$, which is a reduction in expected utility from the first best of $2 - \Psi$, showing the significance of the ICC.

It is clear that for parameter values such that R2 exists, this will dominate R1. Also, by a continuity argument, we can show that there will be parameter values such that R3, R4 and R5 exist in preference to R1 for any given positive level of Ψ. It is possible also to show that there are parameter values such that R1 is chosen in preference to either R3, R4 or R5, but we leave the investigation of this to the interested reader.

John Fender
Peter Sinclair
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