Lecture Notes, Econ 320B. Set # 1. Includes general information about the course.

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1 General Equilibrium and Welfare (Econ 320B)

1.1 Practical Information

Lecturer: Dr. Jensen, Economics Department (JG Smith Building) Room 206.

All information about the course can be found on the course homepage which is located at:

http://socscistaff.bham.ac.uk/jensen/ECON320B.htm

WebCT is *not* used, all you will find in WebCT is the previous link.

1.2 Teaching Material

- Jehle and Reny, Advanced Microeconomic Theory, chapter 5.
- Lecture notes on social choice theory.
- Occasional lecture notes.

All teaching material is distributed at lectures. Except for chapter 5 from Jehle and Reny, all teaching material/handouts can also be downloaded in pdf format from the course homepage.

1.3 Overview

For a detailed reading list which is updated after each lecture, see:

http://socscistaff.bham.ac.uk/jensen/econ320cov.htm

1. General equilibrium in exchange economies. Properties of excess demand functions. Existence of equilibrium. Efficiency.
2. General Equilibrium with production in private ownership economies and under lump-sum transfers.
3. The welfare theorems. The core of a competitive economy.
4. Limitations of classical welfare analysis.

5. Social choice theory.

There's quite a bit of mathematics in this material. I'm sometimes asked for a good "mathematics for economists" book or set of lecture notes. Firstly, try Google "Mathematics for Economists", you'll find several sets of lecture notes (and some of these are very well written and selfcontained). I always recommend the mathematical appendix to Mas-Colell et al. (Microeconomics). You can borrow this in the library and copy the appendix. As for books, I think that Carl P. Simon and Lawrence E. Blume, "Mathematics for Economists" (W W Norton & Co Ltd, 1994) is excellent. It's actually not too expensive considering the fact that it's almost 1000 pages long.

1.4 Method of Assessment

1 problem solving exercises (20 %) + 1 in-class test (20 %) + a two hour examination (60 %) = 100 %

2 Exchange Economies, Basic Notation

- $I \in \mathbb{N}$ is the number of consumers, we always take $I \geq 2$ (two or more consumers).[1] $\mathscr{I} = \{1, \ldots, I\}$ is the *set of consumers*.
- $n \in \mathbb{N}$ is the *number of goods*, so $\{1, \ldots, n\}$ is the set of goods' indices.
- $\mathbf{e}^i = (e_1^i, \ldots, e_i^n) \in \mathbb{R}_+^n$ is consumer i's *endowment vector*. This is also commonly referred to as the *vector of initial resources* or *initial endowment vector*.
- $\mathbf{e} = (\mathbf{e}^1, \ldots, \mathbf{e}^I) \in \mathbb{R}_+^{nI}$ is the *economy's endowment vector*.
- $\mathbf{x}^i = (x_1^i, \ldots, x_n^i) \in \mathbb{R}_+^n$ is a *(consumption) bundle* for consumer i. Collecting the I consumers' bundles we get an *allocation* $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^I) \in \mathbb{R}_+^{nI}$.
- The preference relation of consumer i is denoted by $\succeq^i$.
- The utility function of consumer i is denoted by u^i.

Definition 1 *The utility function* $u^i : \mathbb{R}_+^n \to \mathbb{R}$ **represents** *the preference relation* $\succeq^i$ *if* $\mathbf{x}^i \succeq^i \tilde{\mathbf{x}}^i \Leftrightarrow u^i(\mathbf{x}^i) \geq u^i(\tilde{\mathbf{x}}^i)$, $\mathbf{x}^i, \tilde{\mathbf{x}}^i \in \mathbb{R}_+^n$.

3 Perfect Competition

In a perfectly competitive exchange economy (with private ownership=no government), consumers sell their endowments at the prevailing *(market) prices* $\mathbf{p} = (p_1, \ldots, p_n)$, $p_m > 0$ all m, and use the resulting *income*,

$$\mathbf{p}\mathbf{e}^i = \sum_{k=1}^{n} p_k e_k^i \,,$$

to buy a consumption bundle $\mathbf{x}^i \in \mathbb{R}_+^n$, the *expenditure* of which will be:

$$\mathbf{p}\mathbf{x}^i = \sum_{k=1}^{n} p_k x_k^i$$

Remark 1 *We always require that* $\mathbf{x}^i \in \mathbb{R}_+^n$. *That is to say that the consumption set is assumed to be* $\mathbb{R}_+^n$ *(the set of all non-negative* n*-dimensional vectors).*

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[1] $\mathbb{N}$ denotes the set of natural numbers, *i.e.*, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

The bundle $\mathbf{x}^i$ is *affordable* (or *feasible*) if:

$$\mathbf{p}\mathbf{x}^i \leq \mathbf{p}\mathbf{e}^i \tag{1}$$

Consumers' preferences are represented by utility functions. Consumer i's utility function is:

$$u^i : \mathbb{R}^n_+ \to \mathbb{R}$$

In words, a utility function takes an element in the consumption set $\mathbb{R}^n_+$ and maps it into a real number (an element in $\mathbb{R}$). This real number is the *utility (level)* (sometimes called *level of satisfaction*). The higher utility, the better off the consumer is; so the *objective of the consumer in a perfectly competitive exchange economy* is to choose a consumption bundle $\mathbf{x}^i$ such that (1) is satisfied and given which utility is at its maximum. Formally, the problem is:

$$\begin{aligned} &\max \; u^i(x_1^i, \ldots, x_n^i) \\ &\text{s.t.} \begin{cases} \sum_k p_k x_k^i \leq \sum_k p_k e_k^i \\ x_k^i \geq 0 \text{ for } k = 1, \ldots, n \end{cases} \end{aligned} \tag{2}$$

Definition 2 *A utility function u^i is* **continuous** *if for any convergent sequence $(x^p)_{p=1}^{\infty}$, $x^p \to x$ as $p \to \infty$, it holds that $u^i(x^p) \to u^i(x)$ as $p \to \infty$.*

Definition 3 *A utility function u^i is* **strongly increasing** *if $x_k^i \geq \tilde{x}_k^i$ for all $k = 1, \ldots, n$ with a least one strict inequality, implies that $u^i(\mathbf{x}^i) > u^i(\tilde{\mathbf{x}}^i)$. In words, adding more of one, or more of the goods makes the consumer strictly better off.*

Remark 2 *When the utility function is differentiable, a sufficient condition for it to be strongly increasing is that the partial derivatives are strictly positive. That is,*

$$\frac{\partial u^i(\mathbf{x}^i)}{\partial x_k^i} > 0, \; k = 1, \ldots, n$$

This is how, in assignments, you'd typically be expected to check whether a given utility function is strongly increasing or not.

Definition 4 *A utility function u^i is* **strictly quasi-concave** *if the "better sets" are all strictly convex, i.e., if each of the sets $B(\mathbf{x}^i) \equiv \{\mathbf{y} \in \mathbb{R}^n_+ : u^i(\mathbf{y}) \geq u^i(\mathbf{x}^i)\}$, where $\mathbf{x}^i \in \mathbb{R}^n_+$, is strictly convex.*[2]

[2] A set $B \subseteq \mathbb{R}^n$ is strictly convex if for every pair of elements, $x, \tilde{x} \in B$, $x \neq \tilde{x}$, and any $\lambda \in (0,1)$, the convex combination $\lambda x + (1-\lambda)\tilde{x}$ lies in the interior of B. The set is convex if for every pair of elements, $x, \tilde{x} \in B$, $x \neq \tilde{x}$, and any $\lambda \in [0,1]$, the convex combination $\lambda x + (1-\lambda)\tilde{x}$ lies in B. It is clear that a strictly convex set is also convex.

Remark 3 *There are other ways of defininining strict quasi-concavity (of course these are equivalent to the previous one). Thus if for all* $\mathbf{x}^i, \tilde{\mathbf{x}}^i \in \mathbb{R}^n_+$,

$$u^i(\lambda \mathbf{x}^i + (1-\lambda)\tilde{\mathbf{x}}^i) > \min\{u^i(\mathbf{x}^i), u^i(\tilde{\mathbf{x}}^i)\} \tag{3}$$

u^i *is strictly quasi-concave. Perhaps the most important things to know for assignments (and the exam) is that:*

- *If* u^i *is strictly concave, then* u^i *is strictly quasi-concave.*
- *If* u^i *is concave, then* u^i *is quasi-concave.*

So in order to show that u^i *is strictly quasi-concave, you can try to show that it is strictly concave (which usually is much easier). Thus if* u^i *is twice continuously differentiable, strict concavity of* u^i *(and therefore strict quasi-concavity of* u^i*) holds if the Hesse matrix (the matrix of second partial derivatives) is negative definite. In the case of two goods (*$n = 2$*), it is particularly easy to check that the Hesse matrix is negative definite (the upper left 1-by-1 element of the Hesse matrix must be negative, and the determinant of the Hesse matrix must be positive).*

The following assumption is central to consumer theory:

Assumption 5.1. (JR p.188) *The utility function* $u^i : \mathbb{R}^n_+ \to \mathbb{R}$ *is continuous, strongly increasing, and strictly quasi-concave.*

Under Assumption 5.1. we get Theorem 5.1. (JR p.189). This says that when the price vector is positive in all coordinates (written $\mathbf{p} \gg 0$), the consumer's decision problem ((2) above or (5.2) in JR) has a unique solution:

$$\mathbf{x}^i(\mathbf{p}, \mathbf{p}\mathbf{e}^i)$$

Furthermore, the function $\mathbf{x}^i$ is continuous in $\mathbf{p}$ (for $\mathbf{p} \in \mathbb{R}^n_{++}$).

There are two statements here:

Existence: The consumer's decision problem actually has a solution. There certainly are cases where this would not apply. One such case is when u^i is strictly increasing and one of the prices p_k equals zero (Optional exercise: Explain why!). That a solution exists is good news for our theory: A theory which leads to non-existence is empty, or at least it is unable to explain and predict anything. I'll say a bit more about this at the lectures.

Uniqueness: The consumer's decision problem has at most one solution. So if we ask the consumer what she is going to buy, she'll say "7 apples and 4 oranges", she won't say "well, I might buy 7 apples and 4 oranges, or 2 apples and 6 oranges, or...". Notice in this connection, that when we write the demand function $\mathbf{x}^i(\mathbf{p}, \mathbf{pe}^i)$, we are implicitly employing uniqueness: If there were more than one solution well... We couldn't write $\mathbf{x}^i(\mathbf{p}, \mathbf{pe}^i)$ for *the* solution, right ?

As already mentioned $\mathbf{x}^i(\mathbf{p}, \mathbf{pe}^i)$ is called the **demand function (of consumer i)**. Observe what's going in here: We solve (2) for given prices $\mathbf{p}$ and given endowments $\mathbf{e}^i$. The solution is $\mathbf{x}^i(\mathbf{p}, \mathbf{pe}^i)$ - it *depends* on $\mathbf{p}$ and $\mathbf{e}^i$. Letting $\mathbf{p}$ and $\mathbf{e}^i$ *vary* we then get a function, the demand function.