Aggregate Comparative Statics*

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Abstract

In aggregative games, each player's payoff depends on her own actions and an aggregate of the actions of all the players (for example, sum, product or some moment of the distribution of actions). Many common games in industrial organization, political economy, public economics, and macroeconomics can be cast as aggregative games. In most of these situations, the behavior of the aggregate is of interest both directly and also indirectly because the comparative statics of the actions of each player can be obtained as a function of the aggregate. In this paper, we provide a general and tractable framework for comparative static results in aggregative games. We focus on two classes of aggregative games: (1) aggregative of games with strategic substitutes and (2) "nice" aggregative games, where payoff functions are continuous and concave in own strategies. We provide simple sufficient conditions under which "positive shocks" to individual players increase their own actions and have monotone effects on the aggregate. We show how this framework can be applied to a variety of examples and how this enables more general and stronger comparative static results than typically obtained in the literature.

Keywords: Aggregative games, strategic substitutes, nice games, comparative statics.

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1 Introduction

In aggregative games, each player's payoff depends on her own actions and some aggregate of all players' actions. For example, the Cournot model of oligopoly competition is an aggregative game; each firm's profits depend on its own quantity and total quantity supplied to the market. More generally, the aggregate could be any mapping from the players' action profile to a real number.[1] There is a large and growing literature on aggregative games (see *e.g.* Acemoglu and Jensen (2010), Corchón (1994), Cornes and Hartley (2005), Dubey et al. (2006), Jensen (2010), and Martimort and Stole (2009)). Several game-theoretic models applied in macroeconomics, industrial organization, political economy, and other fields of economics can be cast as aggregative games. These include the majority of the models of competition (Cournot and Bertrand with or without product differentiation), models of (patent) races, models of contests and fighting, models of public good provision, and models with aggregate demand externalities. In many applied problems, the focus is naturally on how the aggregate (*e.g.,* total supply to the market, the price index, probability of innovation, total public good provision) responds to changes in the environment. In addition, comparative statics of individual actions can often be obtained as a function of the aggregate.[2] In this paper, we provide a simple general framework for comparative static analysis in aggregative games. Our approach is applicable to diverse environments that can be cast as aggregative games and enables us to provide sufficient conditions for a rich set of comparative static results. We present two sets of results. First, we focus on aggregative games with strategic substitutes. In games with strategic substitutes, each player's payoff function is supermodular in her own strategy and exhibits decreasing differences in her own strategy and the strategy vector of other players. For aggregative games with strategic substitutes, we establish the following results:

1. Changes in parameters that only affect the aggregate (such as a shift in demand in the Cournot game) always lead to an increase in the aggregate—in the sense that the smallest and the largest elements of the set of equilibrium aggregates increase.

2. Entry of an additional player decreases the (appropriately-defined) aggregate of the existing players.

3. A "positive" idiosyncratic shock, defined as a parameter change that increases the marginal

[1]We discuss games with multi-dimensional aggregates in Section 6.

[2]The fact that a game is aggregative does not imply that players ignore the impact of their strategies on aggregates. When they do so, we say that the equilibrium is *Walrasian Nash* or that the play is Walrasian. Our results are extended to Walrasian Nash equilibria in Section 8. Because in this case there are more more limited "game-theoretic" interactions, the analysis is more straightforward.

payoff of a single player, leads to an increase in that player's strategy and a decrease in the other players' aggregate.

The comparative static results mentioned above are intuitive. But it is surprising that for aggregative games, they hold at the same level of generality as the monotonicity results for supermodular games (in particular, no quasi-concavity or convexity assumptions are needed).

Nevertheless, not all aggregative games exhibit strategic substitutes. The second set of results we present focus on "nice" aggregative games, which are games with payoff functions that are concave (or pseudo-concave) in own strategies, and sufficiently smooth (specifically, twice continuously differentiable).

We prove results that parallel and complement the ones mentioned above for nice aggregative games under an additional assumption, which we refer to as *local solvability.* Under this assumption, which ensures the local invertibility of the "backward reply" mapping described further below, we establish a general monotonicity theorem similar in spirit to the monotonicity results for supermodular games (Milgrom and Roberts (1990), Vives (1990)). This theorem predicts that a positive shock to (one or more of) the players will lead to an increase in the smallest and largest equilibrium values of the aggregate. We also prove that entry of an additional player increases the aggregate and derive more extensive "individual comparative static" results.[3]

Verifying that an aggregative game satisfies the local solvability condition, or alternatively that it is a game of strategic substitutes, is relatively straightforward in any given application. Thus our results provide very easy-to-apply, yet powerful results for applied economics.[4]

An informal summary of our results from both aggregative games with strategic substitutes and from nice aggregative games is that, under a variety of reasonable economic conditions, comparative statics are "regular" (for example, a reduction marginal cost increases output and so on). We next motivate our results and illustrate their scope using two examples. The first focuses on a general class of contest games, which were introduced by Loury (1979) in the context of patent races and then extensively used in the political economy literature starting with the work of Dixit (1987) and Skaperdas (1992). The second is the standard Cournot model.

[3]Though intuitive, robust comparative static results of the form discussed here are generally not true unless the local solvability condition is satisfied. In the Appendix, we provide an example of a simple aggregative game where a positive shock decreases the equilibrium aggregate.

[4] In Section 5 we illustrate this for a number of standard applications of game theory (contests, public good provision, Cournot competition). The simplicity of our approach should be contrasted with the more standard route to comparative statics results that is based on the implicit function theorem (IFT). The application of the IFT in a game-theoretic setting typically requires substantial matrix algebraic computations. Complexity aside, even the most ingenuous applications of the IFT would not yield the main results in this paper, which are global, whereas the IFT uses and delivers local information.

Example 1 Suppose that I players exert effort or invest in their research labs, guns or armies in order to win a contest or fight. Let us denote the strategy of player i (corresponding to effort) by s_i. Then player i's payoff can be written as

$$\pi_i(s_i, s_{-i}) = V_i \cdot \frac{h_i(s_i)}{R + H\left(\sum_{j=1}^{I} h_j(s_j)\right)} - c_i(s_i), \tag{1}$$

where $c_i : S_i \to \mathbb{R}_+$ denotes the cost of effort, $h_1, \ldots, h_I$, and H are strictly increasing functions, and $R \geq 0$. This is an aggregative game, since as (1) makes clear, the payoff of player i depends only on her own action, s_i, and an aggregate of the actions of all other players given by $H\left(\sum_{j=1}^{I} h_j(s_j)\right)$.[5] It can also be easily verified that this is neither a game of strategic complements nor strategic substitutes. In fact, some of the most interesting questions in this context arise because the reduction in the costs of effort for some player i may either increase the effort of a competing player i' (when this player also increases her effort in order to keep up) or decrease it (because the competitor "gives up").

Our approach will enable us to obtain very sharp comparative static results for this class of games, showing that a decline in the cost of effort for player i will always increase her effort, and provide a precise threshold rule, determining whether each of the other players will respond by increasing or decreasing their effort. While recent work, for example, Nti (1997), also presents comparative static results for contest games. However, as explained in the text, our approach can be applied more easily and leads to sharper and more general results (for example, in contrast to Nti (1997), we do not restrict attention to symmetric games).

Example 2 The Cournot model of quantity competition, where I firms each set output $s_i \in [0, \bar{s}_i]$, in order to maximize profits,

$$\pi_i(s,t) = s_i P\left(\sum_{j=1}^{I} s_j\right) - c_i(s_i) \tag{2}$$

is a classical example of an aggregative game. Here P is the inverse demand in this market and c_i is the cost function of firm i. As is well known, under mild conditions this is a game of strategic substitutes and pure strategy equilibria exist even when profit/payoff functions are non-concave and/or strategy sets are non-convex (Novshek (1985), Kukushkin (1994)). To the best of our knowledge, there are, however, no existing comparative statics results that apply to the model at this level of generality (or to the large number of games that share the same abstract features, an example being a public good provision game with a normal private good as returned to in

[5] The precise definition of an aggregate and aggregative games are given in Section 2, where we also show more formally that contests are aggregative games.

Section 2). Our approach applies directly to the Cournot model and provides a complete set of comparative statics results.

Section 5 discusses the application of our general results to these and various other economic models, highlighting both the applicability of the methods we propose and several new results and insights.[6]

At this point, it is worth emphasizing that there is no guarantee in general that intuitive and unambiguous comparative static results should hold in aggregative games. In the Appendix (Section 10.1) we provide a simple example that illustrates how counterintuitive, "perverse," results can arise in simple aggregative games. In this light, a major contribution of our paper is to provide minimal conditions to ensure that such perverse results do not arise. In particular, our first set of theorems shows that such perverse results can not arise in aggregative games with strategic substitutes, and our second set of results establishes that they can be ruled out in nice aggregative games by the local solvability condition mentioned above. The reason we can derive strong unambiguous comparative static results in the two examples discussed above is that in general contests the local solvability condition is satisfied naturally, while the Cournot model is a game of strategic substitutes under natural conditions (and it also satisfies the local solvability conditions under very natural conditions).

In addition to providing minimal conditions for general comparative static results and significantly weakening the conditions that are available in the literature (for example, for models of patent races, contests, and public good provision), our approach is also useful because it highlights the common features that ensure "regular" comparative static results. These results are made possible by the alternative approach we use for deriving comparative static results (the more familiar approaches in the literature rely on the implicit function theorem already discussed above, or lattice-theoretic tools in the case of supermodular games). Our approach can be explained as follows. Consider a general comparative static problem written as

$$\mathbf{A} \cdot \mathbf{ds} = -\mathbf{b} \cdot dt,$$

where $dt \in \mathbb{R}$ is the change in some exogenous variable, $\mathbf{ds} \in \mathbb{R}^M$ designates the induced change in the endogenous variables, $\mathbf{A}$ is an $M \times M$ matrix and $\mathbf{b}$ is an M-dimensional vector. An important question is to specify the conditions under which we can sign $\mathbf{ds}$ "robustly"—meaning without specifying numerical values for the elements of the matrix $\mathbf{A}$ and vector $\mathbf{b}$. Cast in this generality, the conclusion is somewhat depressing: to obtain such results it is necessary to

[6] Another leading example is games in which players make voluntary (private) contributions to the provision of a public good (Section 5.2).

ascertain the sign of the elements of $\mathbf{A}^{-1}$. But even when $\mathbf{A}$ is symmetric negative definite, we can do this only when one of the following two additional (and stringent) conditions hold: (i) when $\mathbf{A}$ is a *Metzler matrix,* that is, it has non-negative off-diagonal elements, or (ii) when $\mathbf{A}$ is a *Morishima matrix*, that is, it becomes a Metzler matrix when we reverse the sign of one or more variables.[7] The only general case where these conditions are satisfied is provided by supermodular games. Since many games that arise in applications are not supermodular, much of the applied literature imposes additional parametric restrictions in the context of specific games to derive comparative statics results. The discussion above highlights that many of these conclusions may not be robust and in general there are no systematic investigations of when the specific conclusions enjoy such robustness.

Our alternative approach is not to impose parametric restrictions, but to exploit the aggregative nature of the games in question and note that what is often of economic interest is not the entire vector $\mathbf{ds}$, but the behavior of the appropriate aggregate (such as the sum of this vector's entries), or just one of its coordinates (the latter corresponds to deriving robust results for a single player as opposed to all players). With this perspective, robust and general comparative static results can be obtained under considerably weaker conditions. Our contribution is to suggest this perspective and show how it can be made operational.

Our paper is related to a number of different strands in the literature. Comparative static results in most games are obtained using the implicit function theorem. The main exception is for supermodular games (games with strategic complements). Topkis (1978, 1979), Milgrom and Roberts (1990) and Vives (1990) provide a framework for deriving comparative static results in such games. These methods do not extend beyond supermodular games.

More closely related to our work, and in many ways its precursor, is Corchón (1994). Corchón (1994) provides comparative static results for aggregative games with strategic substitutes, but only under a fairly restrictive additional condition, which, among other things, implies uniqueness of equilibrium. In contrast, we provide general comparative static results for aggregative games with strategic substitutes without imposing any additional assumptions. We also provide parallel but stronger results for nice aggregative games without strategic substitutes. Another similarity between our paper and Corchón (1994) is that both make use of the so-called *backward reply correspondence* of Selten (1970). In an aggregative game, the backward reply correspondence gives the (best response) strategies of players that are compatible with a given value of the aggregate.[8]

[7] See Bassett et al (1968) and Hale et al (1999).

[8] The first systematic study of aggregative games (German: *aggregierbaren Spiele*) can be found in Selten (1970). After defining aggregative games, Selten proceeds to define what he calls the *Einpassungsfunktion* (Selten (1970), p. 154), that is, the backward reply function of an individual player. As Selten proves, the backward reply correspondence is single-valued (a function) provided that the player's best-response function has slope greater

In a seminal paper, Novshek (1985) used this correspondence to give the first general proof of the existence of pure-strategy equilibria in the Cournot model without assuming quasi-concavity of payoff functions (see also Kukushkin (1994)). Novshek's result has since been strengthened and generalized to a larger class of aggregative games (*e.g.,* Dubey et al. (2006) and Jensen (2010)) and our results on games with strategic substitutes utilize Novshek (1985)'s construction in the proofs.[9] Our results on "nice" aggregative games blend the backward reply approach with the equilibrium comparison results reported in Milgrom and Roberts (1994) and Villas-Boas (1997).[10]

The rest of the paper is organized as follows. Section 2 provides basic definitions. Section 3 provides the general comparative static results for aggregative games with strategic substitutes. Section 4 generalizes and strengthens these results for "nice" aggregative games, which feature payoffs that are continuous and (pseudo-)concave in own strategies. Section 5 shows how the results from Sections 3 and 4 can be used to obtain general characterization results in various applications, including Examples 1 and 2 discussed above. Section 6 discusses how these results can be extended to games with multidimensional aggregates and Section 7 provides additional generalizations of the results presented in Section 4. Section 8 briefly discusses Walrasian Nash equilibria (cf. footnote 2). Section 9 concludes and the Appendix contains the examples and proofs omitted from the text.

2 Basic Definitions

The section introduces the basic definition of an aggregative game used throughout this paper. In this section we also introduce games with strategic substitutes which we return to in the next Section. The concept of a "nice" game is not used until Section 4, hence we postpone its definition until then.

Let $\Gamma = (\pi_i, S_i, T)_{i \in \mathcal{I}}$ denote a noncooperative game with a finite set of players $\mathcal{I} = \{1, \ldots, I\}$, and finite-dimensional strategy sets $S_i \subseteq \mathbb{R}^N$. In addition, $T \subseteq \mathbb{R}^M$ is a set of exogenous parameters with typical element $t \in T$. We will focus on how the set of equilibria of Γ changes in response to changes in t.

than -1. The assumptions imposed by Corchón (1994) imply that the slope of players' best-response functions lie strictly between -1 and 0, so that the backward reply correspondence is both single-valued and decreasing. Neither is necessarily the case in many common games and neither is imposed in this paper.

[9] Novshek's explicit characterization of equilibria is similar to the characterization of equilibrium in supermodular games that uses the fixed point theorem of Tarski (1955). Both of these enable the explicit study of the behavior of "largest" and "smallest" fixed points in response to parameter changes. Tarski's result is used, for example, in the proof of Theorem 6 in Milgrom and Roberts (1990).

[10] More specifically, our proofs repeatedly use that the smallest and largest fixed points of a continuous function from a subset of real numbers into itself will increase when the curve is "shifted up" (see Figure 1 of Villas-Boas (1997) or Figure 2 of Milgrom and Roberts (1994)).

Throughout the rest of the paper we assume that the joint strategy set

$$S \equiv \prod_{j=1}^{I} S_j$$

is compact (in the usual topology) and the payoff functions

$$\pi_i : S \times T \to \mathbb{R}$$

are upper semi-continuous for each $i \in \mathcal{I}$. Let

$$R_i(s_{-i}, t) \equiv \arg\max_{s_i \in S_i} \pi_i(s_i, s_{-i}, t)$$

denote the best response correspondence (with the standard notation $s_{-i} \in S_{-i} \equiv \prod_{j \neq i} S_j$). Given the compactness and upper semi-continuity assumptions, the correspondence R_i is non-empty- and compact-valued, and upper hemi-continuous.

We next define the notion of an *aggregator.*

Definition 1 ***(Aggregator)*** *A mapping $g : S \to \mathbb{R}^K$ (with $K \leq N$) is an* aggregator *if it is continuous, increasing and separable across the players, i.e., if there exists a strictly increasing function $H : \mathbb{R}^K \to \mathbb{R}^K$ and increasing functions $h_i : S_i \to \mathbb{R}^K$ (for each $i \in \mathcal{I}$) such that:*

$$g(s) = H\left(\sum_{j=1}^{I} h_j(s_j)\right). \tag{3}$$

Throughout this paper K is referred to as the *dimension of the aggregate.* For most of the analysis (in particular, until Section 6), we impose $K = 1$, but throughout there are no restrictions on N. In particular, except Corollary 3 in Section 7, none of our results require $N = 1$ (one-dimensional strategy sets). The requirement that g is increasing in s ensures that both g and $-g$ cannot be aggregators for the same game. Naturally, since we can change the order on individual strategies (thus working with $-s_i$ instead of s_i for some i), this requirement is not very restrictive. Common examples, such as the sum of strategies $g(s) = \sum_{j=1}^{I} s_j$, satisfy the definition (with $h_i(s_i) = s_i$ and $H(z) = z$). Two other simple examples are $g(s) = (\alpha_1 s_1^{\beta} + \ldots + \alpha_N s_N^{\beta})^{1/\beta}$, $S \subseteq \mathbb{R}^N$, and $g(s) = \prod_{j=1}^{I} s_j^{\alpha_j}$, $S \subseteq \mathbb{R}^N_{++}$ where $\alpha_j > 0$ (for each j) and $\beta > 0$, which are, respectively, a CES function and a Cobb-Douglas function.[11]

Remark 1 ***(Differentiability and Aggregation)*** In the case of one-dimensional strategy sets, Definition 1 is the standard definition of separability when g is strictly increasing (see, *e.g.,*

[11]In the first case $h_i(s_i) = \alpha_i s_i^{\beta}$ (with $s_i > 0$) and $H(z) = z^{1/\beta}$. In the second $h_i(s_i) = \alpha_i \log(s_i)$ and $H(z) = \exp(z)$ (again with $s_i > 0$).

Gorman (1968)). It can be easily established that when g is twice continuously differentiable, $N = K = 1$, and $I \geq 3$, it is separable if and only if the "marginal rate of transformation" between any two players i and j is independent of the other players' actions; that is,

$$\frac{D_{s_i} g(s)}{D_{s_j} g(s)} = h_{i,j}(s_i, s_j) \text{ for all } s \in S \tag{4}$$

where $h_{i,j} : S_i \times S_j \to R$ is a function of s_i and s_j, but not of any s_q with $q \neq i, j$. More generally, when g is twice continuously differentiable, strictly increasing, and $I \geq 3$, it may be verified that it satisfies Definition 5 if and only if there exist increasing functions $f_i : S_i \times \mathbb{R}^K \to \mathbb{R}^N$ such that for each player $i \in \mathcal{I}$:

$$D_{s_i} g(s) = f_i(s_i, g(s)) \text{ for all } s \in S. \tag{5}$$

When g is increasing (and not necessarily strictly increasing), as is the case in Definition 1, (5) is still implied provided that g is also twice continuously differentiable. This observation will play an important role in Section 4. Clearly, equation (5) also gives an alternative and often very convenient way of verifying that a strictly increasing function g is an aggregator.

Definition 2 ***(Aggregative Game)*** *The game $\Gamma = (\pi_i, S_i, T)_{i \in \mathcal{I}}$ is aggregative if there exists an aggregator $g : S \to \mathbb{R}^K$ and a reduced payoff function*

$$\Pi_i : S_i \times \mathbb{R}^K \times T \to \mathbb{R}$$

for each player i such that

$$\Pi_i \left(s_i, g\left(s \right), t \right) \equiv \pi_i \left(s_i, s_{-i}, t \right). \tag{6}$$

Example 1 (Continued) Recall the payoff functions (1) in a contest. Since both H and the functions h_j are increasing, $g(s) = H(\sum_i h_i(s_i))$ is an aggregator as specified in Definition 1. The game is then aggregative because (6) holds with $\Pi_i \left(s_i, g\left(s \right) \right) \equiv V_i \cdot h_i \left(s_i \right) / \left(R + g\left(s \right) \right) - c_i \left(s_i \right)$ (where the exogenous variables t are suppressed to simplify the notation).

Evidently, an aggregative game is fully summarized by the tuple $((\Pi_i, S_i)_{i \in \mathcal{I}}, g, T)$. Moreover, in an aggregative game, a player i's best-reply correspondence $R_i = R_i(s_{-i})$ can always be expressed as $R_i(s_{-i}, t) = \tilde{R}_i(\sum_{j \neq i} h_j(s_j), t)$, where $\tilde{R}_i$ is a "reduced" best-reply correspondence. In particular, from (3), player i's payoff can be written as a function of the *aggregate of the other players,* $\sum_{j \neq i} h_j(s_j)$, and i's own strategy $s_i \in S_i$, by noting that $\sum_{j \neq i} h_j(s_j) = H^{-1}(Q) - h_i(s_i)$. Given the aggregate $Q = g(s)$, this last observation also allows us to define player i's *backward reply correspondence* as,

$$B_i(Q, t) \equiv \{ s_i \in S_i : s_i \in \tilde{R}_i(H^{-1}(Q) - h_i(s_i), t) \}, \tag{7}$$

which is the set strategies for player i that are best response to the aggregate Q (given parameters t). This backward reply correspondence, which generalizes the concept introduced in Selten (1970), was already motivated in the Introduction, and will play an important role throughout this paper, especially in the proofs.

Another key concept we will use is games with *strategic substitutes.*The payoff function $\pi_i(s_i, s_{-i}, t)$ exhibits *decreasing differences* in s_i and s_{-i} if for all $s_i' > s_i$, the "difference" $\pi_i(s_i', s_{-i}, t) - \pi_i(s_i, s_{-i}, t)$ is a decreasing function in each of the coordinates of $s_{-i} \in S_{-i} \subseteq \mathbb{R}^{N(I-1)}$ (cf. Topkis (1978)). When π_i is twice differentiable, this will hold if and only if $D^2_{s_i s_j}\pi_i(s_i, s_{-i}, t)$ is a non-positive matrix for all $j \neq i$. The payoff function $\pi_i(s_i, s_{-i}, t)$ is *supermodular* in s_i if $\pi_i(s_i \vee \tilde{s}_i, s_{-i}, t) + \pi_i(s_i \wedge \tilde{s}_i, s_{-i}, t) \geq \pi_i(s_i, s_{-i}, t) + \pi_i(\tilde{s}_i, s_{-i}, t)$ for all $s_i, \tilde{s}_i \in S_i$ (and $s_{-i} \in S_{-i}$, $t \in T$). Here $s_i \vee \tilde{s}_i$ ($s_i \wedge \tilde{s}_i$) denotes the coordinatewise maximum (minimum) of the vectors s_i and $\tilde{s}_i$. Naturally, this definition requires that $s_i \vee \tilde{s}_i$ and $s_i \wedge \tilde{s}_i$ are contained in S_i whenever $s_i, \tilde{s}_i \in S_i$, *i.e.*, S_i must be a *lattice*. When strategy sets are one-dimensional, supermodularity as well as the lattice-structure of strategy sets are automatically satisfied, so only decreasing differences remains to be checked. For multidimensional strategy sets, supermodularity holds for twice differentiable payoff functions if and only if $D^2_{s_i^n s_i^m}\pi_i(s_i, s_{-i}, t) \geq 0$ for all $m \neq n$ (where s_i^n and s_i^m denote the nth and mth components of of the strategy vector s_i of player i).

Definition 3 ***(Strategic Substitutes)*** *The game $\Gamma = (\pi_i, S_i)_{i \in \mathcal{I}}$ is a game with strategic substitutes if strategy sets are lattices and each player's payoff function $\pi_i(s_i, s_{-i}, t)$ is supermodular in s_i and exhibits decreasing differences in s_i and s_{-i}.*

Equivalently, we will also say that a game has (or features) strategic substitutes. A game that is both aggregative and has strategic substitutes, is an *aggregative game with strategic substitutes.* Notice that decreasing differences usually is straightforward to verify in aggregative games. In fact, when the aggregator g is a symmetric function there will only be *one* condition to check for each player. For instance, consider an aggregative game with linear aggregator $g(s) = \sum_{j=1}^I s_j$ and one-dimensional strategy sets, so that $\Pi_i\left(s_i, \sum_{j=1}^I s_j, t\right) \equiv \pi_i(s_i, s_{-i}, t)$. If π_i is sufficiently smooth, then decreasing differences is equivalent to nonpositive cross-partials, *i.e.*, $D^2_{s_i s_j}\pi_i = D^2_{12}\Pi_i + D^2_{22}\Pi_i \leq 0$. This immediately implies that if decreasing differences holds for some opponent j, it must hold for all opponents.

Example 2 (Continued) The simplest example of a game with strategic substitutes is the Cournot model $\pi_i(s) = s_i P(\sum_{j=1}^I s_j) - c_i(s_i)$, with a decreasing, concave inverse demand function P (this statement is valid regardless of the cost function c_i).[12]

[12] Assuming differentiability to simplify calculations, the condition for strategic substitutes is $D^2_{s_i s_j}\pi_i(s) =$

Example 3 Another interesting (perhaps less straightforward) example of aggregate of games is the model of private contribution to public good provision introduced in Bergstrom et al. (1986). We discuss this model in greater detail in Section 5.2, and show that it has strategic substitutes if and only if the private good is normal.

Remark 2 ***(Strategic Substitutes and Decreasing Differences in*** s ***and*** Q***)*** Unless players take Q as given (as in Walrasian Nash equilibria discussed in Section 8), there is no exact relationship between strategic substitutes and the condition that $\Pi_i(s_i, Q)$ exhibits decreasing differences in s_i and Q (the latter may be thought of as "strategic substitutes in s_i and the aggregate Q"). For example, suppose that $N = 1$, $g(s) = \sum_{j=1}^{I} s_j$, and assume that payoff functions are twice differentiable. Then the requirement for strategic substitutes is $D^2_{s_i s_q}\Pi_i(s_i, \sum_{j=1}^{I} s_j) = D^2_{12}\Pi_i(s_i, Q) + D^2_{22}\Pi_i(s_i, Q) \leq 0$ where $Q = \sum_{j=1}^{I} s_j$. Decreasing differences in s_i and Q, on the other hand, requires that $D^2_{12}\Pi_i(s_i, Q) \leq 0$. Clearly neither condition implies the other. Provided that $D^2_{22}\Pi_i(s_i, Q) \leq 0$, our strategic substitutes condition is weaker, and in fact, we can have $D^2_{12}\Pi_i(s_i, Q) > 0$ in a game with strategic substitutes.

Finally, we define an equilibrium in the standard fashion.

Definition 4 ***(Equilibrium)*** *Let* $((\Pi_i, S_i)_{i \in \mathcal{I}}, g, T)$ *be an aggregative game. Then* $s^*(t) = (s_1^*(t), \ldots, s_I^*(t))$ *is a (pure-strategy Nash) equilibrium if for each player* $i \in \mathcal{I}$,

$$s_i^*(t) \in \arg\max_{s_i \in S_i} \Pi_i(s_i, g(s_i, s_{-i}^*), t).$$

3 Aggregative Games with Strategic Substitutes

We first present comparative static results for aggregative games with strategic substitutes (defined in Section 2).[13] Strategy sets are allowed to be multi-dimensional for this section's results, but we will assume that the aggregate is one-dimensional (so in terms of Definition 1, $K = 1$ while N is arbitrary). Only the very weak general conditions of Section 2 are needed (upper semi-continuity of payoff functions and compactness of strategy sets). In particular, it is *not* assumed that payoff functions are quasi-concave or that strategy sets are convex. The main result

$P'(Q) + s_i P''(Q) \leq 0$. Clearly this condition holds when P is decreasing and concave.

[13] Note that instead of Definition 3 (supermodularity and decreasing differences), we could equivalently work with *quasi-supermodularity* and the *dual single-crossing property* of Milgrom and Shannon (1994). In fact it is easy to verify from our proofs, that our results will be valid under any set of assumptions that ensure that best-response correspondences are decreasing in the strong set order (*e.g.*, Topkis (1998)). Quasi-supermodularity and dual single-crossing are ordinal conditions and so hold independently of any strictly increasing transformations of the payoff functions. In particular, a payoff function $\pi_i(s_i, s_{-i}, t)$ satisfies the dual single-crossing property in s_i and s_{-i} if for all $s_i' > s_i$ and $s_{-i}' < s_{-i}$, (i) $\pi_i(s_i', s_{-i}) > \pi_i(s_i, s_{-i}) \Rightarrow \pi_i(s_i', s_{-i}') > \pi_i(s_i, s_{-i}')$, and (ii) $\pi_i(s_i', s_{-i}) \geq \pi_i(s_i, s_{-i}) \Rightarrow \pi_i(s_i', s_{-i}') \geq \pi_i(s_i, s_{-i}')$.

of this section is that "regular" comparative statics can be obtained in aggregative games with strategic substitutes without any additional assumptions. Concrete applications of the results can be found in Section 5, in particular that section contains an application to a game where strategy sets are multidimensional illustrating the results' full scope. We begin by noting that an equilibrium will always exist - something which is actually not trivial since no quasi-concavity or convexity assumptions are in force:

Theorem 1 ***(Existence)*** *Let Γ be an aggregative game with strategic substitutes. Then Γ has a Nash equilibrium (i.e., it has at least one pure-strategy Nash equilibrium).*

Proof. See Jensen (2010). ■

Pure-strategy equilibria are not necessarily unique.[14] In general there will be a (compact) *set* $E(t) \subseteq S$ of equilibria for each parameter $t \in T$. When there are multiple equilibria, we focus on the equilibria with the smallest and largest aggregates. The *smallest* and *largest equilibrium aggregates* are defined as

$$Q_*(t) \equiv \min_{s \in E(t)} g(s), \text{ and} \tag{8}$$

$$Q^*(t) \equiv \max_{s \in E(t)} g(s). \tag{9}$$

The following theorem establishes certain important properties of the smallest and largest aggregates, which will be used repeatedly in what follows.

Theorem 2 ***(Smallest and Largest Aggregates)*** *For all $t \in T$, $Q_*(t)$ and $Q^*(t)$ are well defined (i.e., smallest and largest equilibrium aggregates exist). Furthermore the function $Q_* : T \to \mathbb{R}$ is lower semi-continuous, the function $Q^* : T \to \mathbb{R}$ is upper semi-continuous, and thus when there is a unique equilibrium aggregate for all t, $Q_*(t) = Q^*(t)$ is continuous on T.*

Proof. See Section 10.2. ■

Our first substantive result, presented next, addresses the situation where an exogenous parameter $t \in T \subseteq \mathbb{R}$ "hits the aggregator," meaning that it only affects the function g. This result is both of substantive interest and also enables us to prove the subsequent characterization results (in Theorems 4 and 5). More formally, we refer to parameter t as a *shock to the aggregator* (or *aggregate*) when (6) can be strengthened to

$$\Pi_i(s_i, G(g(s), t)) \equiv \pi_i(s, t) \text{ all } i,$$

[14] Uniqueness requires fairly strong additional assumptions. See, for example, Vives (2000), Theorem 2.8, for a result on uniqueness in aggregative games with strategic substitutes and with linear aggregators.

where $g : S \to \mathbb{R}$ designates the aggregator, and $G(g(s), t)$ is continuous, increasing, and separable in s and t (see Definition 1 for the relevant definition of separability). The simplest case would be when the aggregator is linear, so that $\Pi_i \left(s_i, t + \sum_{j=1}^{I} s_j \right) \equiv \pi_i(s, t)$ with $G(g(s), t) = t + \sum_{j=1}^{I} s_j$ and $g(s) = \sum_{j=1}^{I} s_j$. Examples of shocks to the aggregator include a shift in the inverse demand function in the Cournot model (Section 5.3), a change in the discount factor R in a contest/patent race (Section 5.1), or a change in the baseline provision level of the public good $\bar{s}$ in the public good provision model (Section 5.2).

Notice that when t is a shock to the aggregator and t is increased, the marginal payoff of each player *decreases* (provided that marginal payoffs are defined).[15] Hence we would intuitively expect an increasing shock to the aggregator to lead to a *decrease* in the aggregate. The next theorem shows that in an aggregative game with strategic substitutes, this is indeed the case.

Theorem 3 ***(Shocks to the Aggregator)*** *Consider a shock $t \in T \subseteq \mathbb{R}^M$ to the aggregator in an aggregative game with strategic substitutes. Then an increase in t leads to a decrease in the smallest and largest equilibrium aggregates, i.e., the functions $Q_*(t)$ and $Q^*(t)$ are (globally) decreasing in t.*

Proof. See Section 10.3. ■

Though the result in Theorem 3 is intuitive, Example 10.1 (this example was briefly discussed in the Introduction), shows that such results need not hold in simple games, even in simple aggregative games. In Section 5.4 we present an example of an aggregative game with strategic substitutes where a shock (that does not hit the aggregate!) leads to a counter-intuitive equilibrium change in the aggregate.

Also notice that since no concavity or convexity assumptions are required, the conclusion of Theorem 3 could have *never* been reached by use of the implicit function theorem. Similarly, existing results on supermodular games (Topkis (1978), Milgrom and Roberts (1990), Vives (1990)) are obviously of no use when we are dealing with the case of strategic substitutes. In Section 5 we give several illustrations of the usefulness of Theorem 3.

The proof of the theorem exploits the constructive proof of existence of Novshek (1985) (suitably generalized to fit the present framework). This approach provides an explicit description of the largest (and smallest) equilibrium aggregate, allowing us to determine the direction of any change resulting from a shock to the aggregate. We should also add that this approach to comparative statics results is, to the best of our knowledge, new. A major advantage of this approach

[15] By strategic substitutes, agent i's marginal payoff must be decreasing in opponents' strategies and hence, since G is increasing in s and t, an increase in t must lead to a decrease in marginal payoff.

is that it provides *global* results that are valid independently of any differentiability and convexity assumptions.

Theorem 3 also allows us to derive a general result on the effect of "entry", *i.e.*, enables a comparison of equilibria when an extra player is added to the game. The entrant, player $I+1$ when the original game has I players, is (by definition) assigned the "inaction" strategy $\min S_{I+1}$ before entry (*e.g.,* when $S_{I+1} = [0, \bar{s}]$, inaction corresponds to "zero", $s_{I+1} = 0$; for instance, zero production or zero contribution to the provision of a public good). Define the aggregator as $g(s) = g(s_1, \ldots, s_I, s_{I+1})$. Then we have a well-defined aggregative game both *before* and *after* entry; before entry there are I players and s_{I+1} is just a constant, after entry this is an $I+1$ player aggregative game in the usual sense. As elsewhere, here increasing means "either strictly increasing or constant". Thus the entrant may choose "inaction" (zero production in the Cournot model, say) and thus the equilibrium after entry may remain the same.[16]

Theorem 4 ***(Entry)*** *In an aggregative game with strategic substitutes, entry of an additional player will lead to a decrease in the smallest and largest aggregates of the existing players in equilibrium (and a strict decrease if the aggregator g is strictly increasing and the entrant does not choose inaction after entry).*

Proof. This result follows from Theorem 3 by observing that the entry of an additional player corresponds to an increasing shock to the aggregate of the existing players. In particular, let $g(s_1, \ldots, s_I, s_{I+1})$ be the aggregator where $I+1$ is the entrant. Since g is separable, we necessarily have $g(s_1, \ldots, s_I, s_{I+1}) = H(\tilde{g}(s_1, \ldots, s_I), s_{I+1})$ where H and $\tilde{g}$ satisfy the above requirements for a shock to the aggregate (see, for example, Vind and Grodal (2003)). $\blacksquare$

Note that Theorem 4 only shows that the aggregates of the existing players decrease.[17] It is intuitive to expect that the aggregate inclusive of the entrant should increase. But this is not generally true without further assumptions: It may happen that the entrant "crowds out" the existing players' strategies so forcibly that his own positive addition will not make up for the short-fall (see Remark 9 in the proof of Theorem 3 for a detailed description of when entry will decrease the aggregate and when it will not). In the next section, we will present additional assumptions under which entry can be guaranteed to increase the overall aggregate (see Theorem 7).

[16] To strengthen the results to "strictly increasing," one could impose additional boundary conditions.

[17] This is the reason why we do not explicitly write $Q^*(t)$ and $Q_*(t)$. Instead, we could have defined $\tilde{Q}^*(t) \equiv \max_{(s_1,\ldots,s_I)\in E(t)} \tilde{g}(s_1, \ldots, s_I)$ and $\tilde{Q}_*(t) \equiv \min_{(s_1,\ldots,s_I)\in E(t)} \tilde{g}(s_1, \ldots, s_I)$, and the statement would be for $\tilde{Q}^*(t)$ and $\tilde{Q}_*(t)$. But this additional notation is not necessary for the statement or the proof of the theorem.

The next theorem presents what is perhaps our most powerful results for games with strategic substitutes. These can be viewed as strategic substitutes' counterparts to the monotonicity results that are well-known for supermodular games (*e.g.*, Milgrom and Roberts (1990), Vives (1990)). One difference, however, is that with strategic substitutes, the results apply only when shocks are *idiosyncratic*, *i.e.*, to shocks t_i that affect only a *single* player, $i \in \mathcal{I}$. More formally, a change in t_i is an idiosyncratic shock to player i if payoff functions can be written as

$$\begin{aligned} \pi_i(s, t_i) &\equiv \Pi_i(s_i, g(s), t_i), \text{ and} \\ \pi_j(s, t_i) &\equiv \Pi_j(s_j, g(s)) \text{ for all } j \neq i. \end{aligned}$$

Let us also introduce the notion of a *positive shock*.

Definition 5 ***(Positive Shock)*** *Consider the payoff function $\pi_i = \pi_i(s_i, s_{-i}, t_i)$. Then an increase in t_i is a positive shock if π_i exhibits increasing differences in s_i and t.*

It is straightforward to verify that Definition 5 gives the correct notion of "positive shock"; π_i exhibits increasing differences if only if player i's "marginal payoff", $\pi_i(s_i', s_{-i}, t) - \pi_i(s_i, s_{-i}, t)$ for $s_i' > s_i$, is increasing in t. Moreover, as is well known, when π_i is sufficiently smooth, it will exhibit increasing differences in s_i and t if and only if the cross-partial is nonnegative, *i.e.,* $D^2_{s_i t}\pi \geq 0$ for all s and t. The single-crossing property may replace increasing differences in the previous definition without changing any of our results. We also define smallest and largest equilibrium strategies for player i analogously to the smallest and largest equilibrium aggregates.

Theorem 5 ***(Idiosyncratic Shocks)*** *Let t_i be a positive idiosyncratic shock to player i. Then an increase in t_i leads to an increase in the smallest and largest equilibrium strategies for player i, and to a decrease in the associated aggregates of the remaining players (which are, respectively, the largest and smallest such aggregates).*

Proof. See Section 10.4. ■

A simple corollary to Theorem 5 also characterizes the effects of a positive shock on payoffs.

Corollary 1 ***(Payoff Effects)*** *Assume in addition to the conditions of Theorem 5 that all payoff functions are decreasing [respectively, increasing] in opponents' strategies and that player i's payoff function is increasing [respectively, decreasing] in the idiosyncratic shock t_i. Then an increase in t_i increases [respectively, decreases] player i's payoff in equilibrium and decreases [respectively, increases] the payoff of at least one other player.*

Proof. For player i, $\pi_i(s'_i, g(s'), t') \leq \pi_i(s'_i, g(s'_i, s''_{-i}, t'') \leq \pi_i(s''_i, g(s''), t'')$. Since the strategy of some player j (for $j \neq i$) decreases, we must have $\sum_{k\neq j} h_k(s'_k) \leq \sum_{k\neq j} h_k(s''_k)$. Consequently, $\pi_j(s''_j, g(s'')) \leq \pi_j(s''_j,\, g(s''_j, s'_{-j})) \leq \pi_j(s'_j, g(s'))$. ■

4 Nice Aggregative Games

We now extend the framework of the previous section to aggregative games without strategic substitutes. For this purpose, we focus on "nice" games where payoff functions are differentiable and concave (or pseudo-concave) in own strategies.[18] As in the previous section, we focus on games where the aggregate is one-dimensional, hence we can speak of and study the smallest and largest equilibrium aggregates characterized as a function of the exogenous parameters in Theorem 2. Our main result (Theorem 6) establishes that in a nice aggregative game, the largest and smallest equilibrium aggregates increase with positive shocks whenever the *local solvability condition* holds (Definition 7 below). An application to *contests* can be found in Section 5.1 (contests are neither games of strategic substitutes or strategic complements). Although we use first-order conditions in our analysis, the results in this section belong to the class of *global comparative statics theorems* alongside the results of the previous section and parallel results for supermodular games (*e.g.*, Milgrom and Roberts (1990), Vives (1990)). In particular, the results could *never* be reached by a standard application of the implicit function theorem since the implicit function theorem could never predict anything *but* the continuous dependence of endogenous variables on exogenous ones.[19]

The following definition introduces the notion of "nice" aggregative games formally. When strategy sets are one-dimensional ($N = 1$), the boundary condition featured in the definition can be dispensed with if the local solvability condition is strengthened (Definition 8). As explained in Remark 4, the boundary condition is implied by standard Inada-type boundary conditions (whether or not $N = 1$).

Definition 6 ***(Nice Aggregative Games)*** *An aggregative game* Γ *is said to be a* nice aggrega*tive game if the aggregator* g *is twice continuously differentiable, each strategy set is compact and convex, and every payoff function* π_i *is twice continuously differentiable, and pseudo-concave in*

[18] Weinstein and Yildiz (2008) use a similar definition of a "nice game," except that they also impose one-dimensional strategies.

[19] To be a bit more specific, a global comparative statics result applies whether or not changes in parameters are "small", and it applies equally if the equilibrium aggregate (and consequently the associated strategies) changes discontinuously with a continuous change in parameters - something which to be sure, may easily happen under our general conditions. A third and perhaps less obvious aspect of our results is that these, as the results in the previous section, yield statements about the largest and smallest equilibrium aggregates. So there is an element of "equilibrium selection" working to our advantage in the background. The implicit function theorem will "miss" any such global information making it in effect useless for the present purpose.

the player's own strategies. Finally, when $s_i \in \partial S_i$ (with ∂S_i denoting the boundary of the strategy set S_i) and $(v - s_i)^T D_{s_i}\pi_i(s,t) \leq 0$ for all $v \in S_i$, then $D_{s_i}\pi_i(s,t) = 0$. That is, the first-order conditions $D_{s_i}\pi_i(s,t) = 0$ are required to hold whenever a boundary strategy for player i is a (local) best response.

Remark 3 ***(Pseudo-concavity)*** Recall that a differentiable function π_i is pseudo-concave (Mangasarian (1965)) in s_i if for all $s_i, s_i' \in S_i$:

$$(s_i' - s_i)^T D_{s_i}\pi_i(s_i, s_{-i}, t) \leq 0 \;\Rightarrow\; \pi_i(s_i', s_{-i}, t) \leq \pi_i(s_i, s_{-i}, t).$$

Naturally, any concave function is pseudo-concave. Pseudo-concavity implies that the first-order conditions $D_{s_i}\pi_i(s,t) = 0$ are sufficient for s_i to maximize π_i given s_{-i} and t. That first-order conditions are sufficient for a maximum is what we use in the proofs. Pseudo-concavity is not a necessary condition for this to hold. For example, if $N = 1$ and $D_{s_i}\pi_i(s,t) = 0 \Rightarrow D^2_{s_i s_i}\pi_i(s,t) < 0$, it is easy to see that the first-order condition will be sufficient for a maximum (and in fact, that the maximum will be unique). Quasi-concavity (or even strict quasi-concavity) does not imply the sufficiency of first-order conditions for a maximum in general.

Remark 4 ***(Inada-Type Boundary Conditions)*** Note also that the boundary condition in Definition 6 does not rule out best responses on the boundary of a player's strategy set, ∂S_i. Instead, it simply requires first-order conditions to be satisfied whenever a local best response is on the boundary. Consequently, this boundary condition is weaker than the standard "Inada-type" conditions ensuring that best responses always lie in the interior of strategy sets (since when best responses never lie on the boundary, first-order conditions vacuously hold for best responses *on* the boundary).[20]

As is well-known, the concavity or pseudo-concavity conditions ensure that best response correspondences are convex-valued (they are also upper hemi-continuous as mentioned at the beginning of Section 2). The existence of a pure-strategy Nash equilibrium therefore follows by Kakutani's fixed point theorem.[21] None of the assumptions in this section guarantee uniqueness,

[20] However, all boundary conditions cannot be dispensed with. To see this, consider an N-dimensional game, $N > 1$ (with each player having N-dimensional strategy sets) without any interior best responses. The boundary of this N-dimensional game can then be mapped bijectively into an $N-1$-dimensional game. But since first-order conditions never have to hold in the N-dimensional game, the local solvability condition below (Definition 7) would never have to hold. In effect, the $N-1$-dimensional "reduction" is therefore unrestricted and consequently, no general results can be derived for such a game.

[21] Without convex best response correspondences, a Nash equilibrium may fail to exist in an aggregative game (unless the game also features strategic substitutes or complements). See Jensen (2010), Example 5 for an example of an aggregative game where a pure-strategy Nash equilibrium fails to exist even though strategy sets are one-dimensional, convex, and there are only two players.

however. We therefore deal with the possible multiplicity of equilibria as in the previous section and study the behavior of the smallest and largest equilibrium aggregates, $Q_*(t)$ and $Q^*(t)$. Theorem 2 from the previous section still applies so the smallest and largest equilibrium aggregates exist and are, respectively, lower and upper semi-continuous, in t.

We next introduce *the local solvability condition*, which will play a central role in our analysis in this section. Let us simplify notation by defining $D_1\Pi_i(s_i, Q, t) \equiv D_{s_i}\Pi_i(s_i, Q, t)$ and $D_2\Pi_i(s_i, Q, t) \equiv D_Q\Pi_i(s_i, Q, t)$. Using the fact that g is twice continuously differentiable, the marginal payoff for player i can then be expressed as (here $f_i(s_i, g(s)) = D_{s_i}g(s)$, cf. Remark 1):

$$D_{s_i}\pi_i(s,t) = D_1\Pi_i(s_i, g(s), t) + D_2\Pi_i(s_i, g(s), t)f_i(s_i, g(s)). \tag{10}$$

Equation (10) shows us that in an aggregative game, a player's marginal payoff can always be written as a function of the player's own strategy s_i and the aggregate $g(s)$. To make this feature of an aggregative game operational, define a function $\Psi_i : S_i \times \mathbb{R} \times T \to \mathbb{R}^N$ by:

$$\Psi_i(s_i, Q, t) \equiv D_1\Pi_i(s_i, Q, t) + D_2\Pi_i(s_i, Q, t)f_i(s_i, Q). \tag{11}$$

Note that this function contains the same information as (10), though it also enables us to separate the direct and indirect effects of a change in s_i on the player's marginal payoff: the direct effect corresponds to a change in Ψ_i holding Q constant while the indirect effect captures the marginal payoff effect of an isolated change in the aggregate. Differentiating Ψ_i with respect to s_i yields an $N \times N$ matrix $D_{s_i}\Psi_i(s_i, Q, t) \in \mathbb{R}^{N\times N}$ which precisely measures the mentioned direct marginal payoff effect. The determinant of this matrix is denoted by $|D_{s_i}\Psi_i(s_i, Q, t)| \in \mathbb{R}$. If strategy sets are one-dimensional we simply have $|D_{s_i}\Psi_i(s_i, Q, t)| = D_{s_i}\Psi_i(s_i, Q, t) \in \mathbb{R}$. We are now ready to define the local solvability condition and discuss its intuitive meaning.

Definition 7 ***(Local Solvability)*** *Player $i \in \mathcal{I}$ is said to satisfy the local solvability condition if $|D_{s_i}\Psi_i(s_i, Q, t)| \neq 0$ whenever $\Psi_i(s_i, Q, t) = 0$ (for $s_i \in S_i$, $Q \in \{g(s) : s \in S\}$).*

As mentioned above, the following stronger version of local solvability will allow us to dispense with any boundary conditions if strategy sets are one-dimensional.

Definition 8 ***(Uniform Local Solvability)*** *When $S_i \subseteq \mathbb{R}$, player $i \in \mathcal{I}$ is said to satisfy the uniform local solvability condition if $D_{s_i}\Psi_i(s_i, Q, t) < 0$ whenever $\Psi_i(s_i, Q, t) = 0$ (for $s_i \in S_i$, $Q \in \{g(s) : s \in S\}$).*

To interpret these conditions, notice first that if the player actually takes the aggregate as given (meaning that she does not take her direct effect on the aggregate $g(s)$ into account when maximizing payoff), uniform local solvability is nothing but strict concavity of the payoff function at any point of maximum (clearly $\Psi_i(s_i, Q, t) = 0$ is the same as saying that the first-order conditions hold with equality given s_i and Q). Indeed, when the player does not take her influence on the aggregate into account, the direct effect as measured by $D_{s_i}\Psi_i$ will simply equal the player's marginal payoff. One way to think of local solvability is thus as a kind of "compensated strict concavity" (or better, "compensated strict-second order conditions"), the compensation being of course with respect to the indirect effect of the aggregate which is held fixed when Ψ_i is differentiated with respect to s_i. Obviously, such compensated/indirect strict concavity does not imply that the game features strategic substitutes or complements as the following examples also makes clear.[22]

Example 2 (Continued) In the Cournot model $\Psi_i(s_i, Q) = P(Q) + s_i P'(Q) - c_i'(s_i)$. Hence the uniform local solvability condition will hold if $D_{s_i}\Psi_i(s_i, Q) = P'(Q) - c_i''(s_i) < 0$ whenever $P(Q) + s_i P'(Q) - c_i'(s_i) = 0$. Clearly this holds if costs are convex and inverse demand is strictly decreasing, which certainly does not imply the strategic substitutes condition $P'(Q)+s_i P''(Q) \leq 0$ (see footnote 12). In fact, as may be verified a different sufficient condition for uniform local solvability when c_i is strictly increasing is that $\frac{s_i c_i''(s_i)}{c_i'(s_i)} > 1$. This condition depends *only* on the cost function c_i, whereas the condition for strategic substitutes depends *only* on the inverse demand function P. So here we are actually looking at fully independent conditions for one or the other to hold.[23]

Example 3 (Continued) We show in Section 5.2 that the model of private contributions to public good provision satisfies the uniform local solvability condition if and only if the public good is strictly normal. In contrast, this will be a game of strategic substitutes if and only if the private good is normal (see the first part of Example 3 in Section 2).

Example 1 (Continued) Contests also satisfy the uniform local solvability condition under very weak and natural conditions introduced in Proposition 1 in Section 5. Contests are neither games

[22] Interestingly, the converse statement is true for games with strategic complements, a linear aggregator ($g(s) = \sum_i s_i$), and strictly concave payoff functions. To see this, note that it always holds that $D^2_{s_i s_i}\pi_i = D_{s_i}\Psi_i + D_Q\Psi_i$. The game has strategic complements if and only if $D_Q\Psi_i \leq 0$ everywhere. So if payoff functions are strictly concave (or if they are merely concave and the game has strict strategic complements, $D_Q\Psi_i < 0$), it must hold that $D_{s_i}\Psi_i < 0$ which is uniform local solvability.

[23] It is worth noting that the condition $P'(Q) - c_i''(s_i) < 0$ is "one half" of Hahn (1962)'s conditions for local stability of Cournot equilibrium (see Vives (1990), Chapter 4 for an extensive discussion of this and related conditions). The "other half" is the condition for strategic substitutes just stated. As mentioned by Corchón (1994) (p. 156), Corchon's "strong concavity condition" reduces precisely to the two Hahn conditions in the Cournot model (except that strategic substitutes is strengthened to strict strategic substitutes). As a consequence, the results to follow will be seen to generalize Corchón (1994)'s results for the Cournot model (we return to this issue in section 5.3).

of strategic substitutes or strategic complements.

Thinking of the local solvability condition somewhat more formally, the requirement is that the determinant of $D_{s_i}\Psi_i$ is nonzero on the subspace where $\Psi_i = 0$. The term refers to the fact that if this determinant were equal to zero, there would not be a unique solution to a set of equations of the form $D_{s_i}\Psi_i \cdot a = b$ (again defined on the subspace where $\Psi_i = 0$), where a and b are N-dimensional vectors. This type of equation arises when constructing the backward reply correspondences for the purposes of comparative static analysis. This discussion also motivates the term "local solvability condition". The local solvability condition's generality and role for comparative statics analysis is discussed further in two remarks at the end of this section.

We next introduce the appropriate notion of positive shocks for nice games. Since the definition aims for maximum generality, we mention already here that if a player i's strategy set is a lattice, his payoff function is supermodular in s_i, and exhibits increasing differences in s_i and t, then the shock will be positive.[24] Thus this section's notion of a positive shock is weaker than that of Section 3. But in general, a positive shock need not satisfy these standard conditions from the theory of monotone comparative statics - in particular, it is irrelevant for our results here whether or not strategy sets are lattices.

Because the aggregator is separable it can be written as $g(s) = H\left(\sum_{j=1}^{I} h_j(s_j)\right)$ where $h_i : S_i \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ (cf. Definition 1). It is clear that the term $h_i(s_i)$ fully "captures" agent i's effect on the aggregate. Intuitively, our generalized definition of a positive shock requires that an increase in the exogenous variable leads to an increase in the term $h_i(s_i)$ and thus increases the aggregate given everyone else's strategies. In comparison, our previous definition, Definition 5, made the stronger requirement that a player's (smallest and largest) strategy should increase with the exogenous variable.

Definition 9 ***(Positive Shocks)*** *A change in the parameter vector t is a* positive shock *to player i if the largest and smallest selections from this player's "composite" best responds correspondence $h_i(R_i(s_{-i}, t))$ are increasing in t. That is, consider $t < t' \in T$ and let $\overline{h_i \circ r_i}(s_{-i}, t)$ and $\underline{h_i \circ r_i}(s_{-i}, t)$ be the maximal and minimal elements of $h_i(R_i(s_{-i}, t)) \subseteq \mathbb{R}$. Then t is a positive shock to player i if and only if $\overline{h_i \circ r_i}(s_{-i}, t) \leq \overline{h_i \circ r_i}(s_{-i}, t')$ and $\underline{h_i \circ r_i}(s_{-i}, t) \leq \underline{h_i \circ r_i}(s_{-i}, t')$ for all $s_{-i} \in S_{-i}$.*

Our first result in this section characterizes the comparative statics of the aggregate. It is similar in spirit to Theorem 3 from the previous section, except that the shock is not restricted

[24] The same observation applies if the payoff function instead satisfies the weaker quasi-supermodularity and single-crossing conditions of Milgrom and Shannon (1994).

to "hit the aggregate". Comparison can also be made with Theorem 5 of the previous section, but unlike that Theorem, the following result applies whether or not the shock hits only one of the players.

Theorem 6 ***(Aggregate Comparative Statics)*** *Consider a nice aggregative game where each player's payoff function satisfies the local solvability condition. Then a positive shock $t \in T$ leads to an increase in the smallest and largest equilibrium aggregates, i.e., the functions $Q_*(t)$ and $Q^*(t)$ are (globally) increasing in t.*

The same results apply without any boundary conditions on payoff functions when $N = 1$ and the uniform local solvability condition is satisfied.

Proof. See Section 10.5 ■

Our next result is parallels Theorem 4 of the previous section, but now makes a predictive statement about the overall aggregate after entry of an additional player (instead of the aggregate of the strategies of existing players). Let us define $0 \in S_i$ to stand for "inaction". As in the previous section, the convention is that $I + 1$th player takes this action before entry.

Theorem 7 ***(Entry)*** *Consider a nice aggregative game where each player's payoff function satisfies the local solvability condition. Let $Q_*(I)$ and $Q^*(I)$ denote the smallest and largest equilibrium aggregates in a game with $I \in \mathbb{N}$ players where $S_i \subseteq \mathbb{R}_+^N$ and $0 \in S_i$ for all $i \in \mathcal{I}$. Then for any $I \in \mathbb{N}$, $Q_*(I) \leq Q_*(I+1)$ and $Q^*(I) \leq Q^*(I+1)$, i.e., entry increases the smallest and largest aggregates in equilibrium. Moreover, if the aggregator g is strictly increasing and the entrant chooses a non-zero strategy following entry, the above inequalities are strict.*

The same results apply without any boundary conditions on payoff functions when $N = 1$ and the uniform local solvability condition is satisfied.

Proof. See Section 10.6. ■

Finally, our third result characterizes the comparative statics of individual strategies. It is useful to note Theorem 8 is the first (and only) result among those presented in this and the previous section that directly depend on the implicit function theorem. As such it is a purely *local* result and also requires that the equilibrium strategy studied is interior.

Theorem 8 ***(Individual Comparative Statics)*** *Consider a nice aggregative game where each player's payoff function satisfies the local solvability condition (or the case with $N = 1$ without boundary conditions but with the local solvability condition). Consider player i's equilibrium* *strategy $s_i^*(t)$ associated with the smallest (or largest) equilibrium aggregate at some equilibrium $s^* = s^*(t)$ given $t \in T$. Assume that the equilibrium s^* lies in the interior of S and that t is a positive shock. Then the following results hold:*

- *$s_i^*(t)$ is (coordinatewise) locally increasing in t provided that*

$$-[D_{s_i}\Psi_i(s_i^*, g(s^*), t)]^{-1} D_Q \Psi_i(s_i^*, g(s^*), t) \geq 0$$

- *Suppose that the shock t does not directly affect player i (i.e., $\pi_i = \pi_i(s)$). Then the sign of each element of the vector $D_t s_i^*(t)$ is equal to the sign of each element of the vector $-[D_{s_i}\Psi_i(s_i^*, g(s^*))]^{-1} D_Q \Psi_i(s_i^*, g(s^*))$. In particular, $s_i^*(t)$ will be (coordinatewise) locally decreasing in t whenever:*

$$-[D_{s_i}\Psi_i(s_i^*, g(s^*))]^{-1} D_Q \Psi_i(s_i^*, g(s^*)) \leq 0$$

Proof. By the implicit function theorem, we have

$$D_{s_i}\Psi_i(s_i, Q, t) ds_i = -D_Q \Psi_i(s_i, Q, t) dQ - D^2_{s_i t} \Pi_i(s_i, Q, t) dt.$$

The results follow from this observation and the fact that Q increases with t (where Q is either the smallest or largest equilibrium aggregate). ■

We end this section with two remarks the further characterizes the scope and content of the local solvability condition for comparative statics.

Remark 5 ***(Weaker Conditions)*** Some version of the local solvability condition cannot be dispensed with for the previous results. Example 10.1 in the Introduction shows the possibility of perverse comparative statics when the local solvability condition does not hold (see also Section 5.4). Nevertheless, the results presented in this section continue to hold under weaker conditions. In particular, the following generalization would be sufficient, though the condition in Definition 7 is easier to state and verify. The alternative condition is as follows: when $\Psi_i(s_i, Q, t) = 0$, there should be open neighborhoods $\mathcal{N}_{s_i}$ and $\mathcal{M}_Q$ of s_i and Q, respectively, and a continuous map $b_i : \mathcal{M}_Q \to \mathcal{N}_{s_i}$ such that for each $\hat{Q} \in \mathcal{M}_Q$, $b_i(\hat{Q})$ is the unique solution to $\Psi_i(s_i, \hat{Q}, t) = 0$ in $\mathcal{N}_{s_i}$. This implies that that the first-order condition $\Psi_i(s_i, Q, t) = 0$ admits a local solution in s_i as a function of Q. Naturally, in view of the implicit function theorem, this weaker condition follows from our local solvability condition in Definition 7. Other alternatives to the local solvability condition are discussed in Section 7.

Remark 6 ***(Ordinality)*** It is useful to note that the local solvability condition is ordinal in several ways: it is independent of any strictly increasing transformation of payoff functions as well as any strictly increasing transformation of the aggregator. Also, the local solvability condition does not depend on the choice of the coordinate system (diffeomorphic transformations of the strategy sets). Let us verify these claims in turn. Firstly, local solvability holds for the payoff function $\pi_i(s,t)$ if and only if it holds for $\Phi(\pi_i(s,t))$ where $\Phi : \mathbb{R} \to \mathbb{R}$ is any strictly increasing and twice continuously differentiable function, with derivative denoted by Φ' (where differentiability is needed here to ensure that the transformed payoff function is also twice continuously differentiable). In particular, for all s_i' and Q', we have that

$$\Psi_i(s_i', Q', t) = 0 \Leftrightarrow \Phi'(\Pi_i(s_i, Q, t))\Psi_i(s_i, Q, t) = 0.$$

Ordinality of the local solvability condition follows if $|D_{s_i}\Psi_i(s_i', \ Q')| \neq 0$ implies $|D_{s_i}[\Phi'(\Pi_i(s_i', Q', t))\Psi_i(s_i', Q', t)]| \neq 0$. This is true since, when $\Psi_i(s_i', Q', t) = 0$, $|D_{s_i}[\Phi'(\Pi_i(s_i', Q', t))\Psi_i(s_i', Q', t)]| = \Phi'(\Pi_i(s_i', Q', t))|D_{s_i}\Psi_i(s_i', Q')|$.

That local solvability is independent of any strictly increasing transformation of the payoff function immediately implies that if it holds with the aggregator $g(s)$, then it holds with the aggregator $\tilde{g}(s) = f(g(s))$ where $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and differentiable function. To see this simply note that $\Pi_i(s_i, g(s)) = \Pi_i(s_i, f^{-1}(\tilde{g}(s)))$ where $\tilde{g}(s) = f(g(s))$. Denoting the new aggregate by $\tilde{Q} = \tilde{g}(s)$, it is clear that $\Psi_i(s_i, \tilde{Q}) = \Psi_i(s_i, Q)$. Evidently then $|D_{s_i}\Psi_i(s_i, Q)| = |D_{s_i}\Psi_i(s_i, \tilde{Q})|$ and the conclusion follows.

Finally, the local solvability condition is a "coordinate free" assumption in the sense that we may replace each strategy vector s_i by a transformed vector $\tilde{s}_i = \psi_i(s_i)$ where $\psi_i : \mathbb{R}^N \to \mathbb{R}^N$ is a diffeomorphism. The local solvability condition then holds for the original strategies if and only if it holds for the transformed ones. To see this notice that given such transformations, the payoff function of player i becomes $\pi_i(\psi_i^{-1}(\tilde{s}_i), \psi_{-i}^{-1}(\tilde{s}_{-i}), t)$, where $\psi_{-i}^{-1} = (\psi_j^{-1})_{j \neq i}$. Local solvability requires that,

$$D\psi_i^{-1}(\tilde{s}_i)\Psi_i(\psi_i^{-1}(\tilde{s}_i), Q) = 0 \Rightarrow |D_{\tilde{s}_i}[D\psi_i^{-1}(\tilde{s}_i)\Psi_i(\psi_i^{-1}(\tilde{s}_i), Q)]| \neq 0.$$

Since $D\psi_i^{-1}(\tilde{s}_i)\Psi_i(\psi_i^{-1}(\tilde{s}_i), Q) = 0 \Leftrightarrow \Psi_i(\psi_i^{-1}(\tilde{s}_i), Q) = 0$ ($D\psi_i^{-1}(\tilde{s}_i)$ is a full rank matrix), we have that

$$|D_{\tilde{s}_i}[D\psi_i^{-1}(\tilde{s}_i)\Psi_i(\psi_i^{-1}(\tilde{s}_i), Q)]| = |D\psi_i^{-1}(\tilde{s}_i)D_{s_i}\Psi_i(\psi_i^{-1}(\tilde{s}_i), Q)[D\psi_i^{-1}(\tilde{s}_i)]^T|.$$

But it is clear that the latter determinant will be non-zero if and only if $|D_{s_i}\Psi_i(\psi_i^{-1}(\tilde{s}_i), Q)| = |D_{s_i}\Psi_i(s_i, Q)| \neq 0$.

5 Applying the Theorems

In this section, we return to the examples discussed so far (contests, Cournot competition and private contributions to public goods) and show that our methods allow very general comparative static results in these models. We also illustrate how our results can be applied to games with multidimensional strategies using a model of technology choice in oligopolistic competition.

5.1 Models of Contests and Fighting

Recall that general contests introduced in Example 1 in the Introduction. The payoff function of participant i can be written as

$$\pi_i\left(s_i, s_{-i}\right) = V_i \cdot \frac{h_i\left(s_i\right)}{R + H\left(\sum_{j=1}^{I} h_j\left(s_j\right)\right)} - c_i\left(s_i\right), \tag{12}$$

where s_i denotes agent i's *effort*, $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ for each $i \in \mathcal{I}$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$. The formulation chosen here is very general, and allows not just for standard contests (where often R is taken equal to zero), but also includes models of rent-seeking, as well as patent races in the spirit of Loury (1979), Dixit (1987) and Skaperdas (1992).

As mentioned in the Introduction, contests generally feature neither strategic substitutes nor complements. Therefore, the results in Section 3 do no apply, nor do any of the well-known results on supermodular games mentioned in the Introduction. In this case, the most obvious strategy for deriving comparative static results is to use the implicit function theorem. This is indeed what most of the literature does. Unfortunately, the implicit function theorem approach yields no unambiguous conclusions unless we make additional, strong assumptions. For this reason, previous treatments have restricted attention to special cases of the above formulation. For example, Tullock (1980) studied two-player contests, while Loury (1979) focused on symmetric contests with (ad hoc) stability conditions. The most general comparative statics results available in the literature are to our knowledge those of Nti (1997) who assumes that agents are identical (the game is symmetric), that $H = id$ (the identity function), that $h_i = h$ for all i and concave (a symmetric, concave contest success function), and that costs are linear ($c_i(s_i) = \bar{c}s_i$ for some constant $\bar{c} > 0$).

Using the results of Section 4, we can establish considerably more general and robust results on this important class of models. In particular, no symmetry assumptions are imposed what so ever.[25]

[25] Since we do not assume concavity of payoff functions, the following proposition also generalizes the existence result of Szidarovszky and Okuguchi (1997).

Proposition 1 *Consider the contest games introduced in Example 1 and suppose that H is convex, h_i and c_i are strictly increasing, and that the following condition holds:*

$$\frac{h_i''(s_i)}{h_i'(s_i)} \leq \frac{c_i''(s_i)}{c_i'(s_i)} \text{ for all } s_i \in S_i.$$

Then there exists a (pure-strategy) Nash equilibrium. Furthermore:

1. *The smallest and largest aggregate equilibrium efforts are increasing in any positive shock (e.g., a decrease in R or an increase in V_i for one or more players).*
2. *Entry of an additional player increases the aggregate equilibrium effort.*
3. *There exists a function $\eta : \mathbb{R} \to \mathbb{R}$ such that the changes in parts 1 or 2 above are associated with an increase in the effort of player $i \in \mathcal{I}$ and the corresponding equilibrium aggregate Q^* provided that i is "dominant" in the sense that $h_i(s_i^*) \geq \eta(Q^*)$. Conversely, if i is "not dominant", i.e., $h_i(s_i^*) < \eta(Q^*)$, then the changes in parts 1 and 2 decrease player i's effort provided that the shock does not affect this player directly (e.g., corresponding to a decrease in another player's costs).*

Proof. The proof simply involves verifying the uniform local solvability condition and applying the results from Section 4. The details are given in Section 10.7 in the Appendix. ■

Proposition 1 can also be extended to the case in which H is not convex. Convexity of H ensures that the first-order condition $D_{s_i}\pi_i(s_i, s_{-i}) = 0$ is sufficient for a maximum, but it is *not* necessary for this conclusion. Observe also that the conditions of Proposition 1 are satisfied if H is the identity function, c_i is convex, and h_i is concave.[26] Szidarovszky and Okuguchi (1997) prove that these conditions imply uniqueness of equilibrium provided that $R = 0$ in (12).[27] Such uniqueness is not necessary or assumed in Proposition 1. In addition, Proposition 1 also

[26]The proof of Proposition 1 shows that the function η in part 3 is given by

$$\eta(Q^*) \equiv \left[\frac{2H'\left(H^{-1}(Q^*)\right)}{(R+Q^*)} - \frac{H''\left(H^{-1}(Q^*)\right)}{H'\left(H^{-1}(Q^*)\right)}\right]^{-1}.$$

Therefore, when, for example, $H = h_i = \text{id}$ (the identity function), and $R = 0$, we have $\eta(Q^*) = Q^*/2$, and so player i is "dominant" if and only if $s_i^* \geq Q^*/2$. In the standard interpretation of a contest, this means that she is dominant when her probability of winning the prize is greater than 1/2—i.e., when she is a *favorite* in the terminology of Dixit (1987). However, this favorite-to-win interpretation does not necessarily apply for more general games covered by Proposition 1. We therefore use the term "dominant" rather than "favorite".

[27]More recently, Cornes and Hartley (2005) have proposed a very nice and simple proof of this result based on what they refer as "share functions". Although Cornes and Hartley do not consider comparative statics, their "share function" approach could be used to establish results similar to the results in Proposition 1 under these stronger assumptions if, *in addition*, one also imposed that $R = 0$ in (12). $R = 0$ amounts to assuming no discounting in patent races and "no wastage" in contests, and is thus quite restrictive.

When $R > 0$, the "share function" approach cannot be used to derive robust comparative statics. The reason covers important cases where h_i is not concave. For example, Hirshleifer (1989) proposes the *logit specification* of the contest success function, with $H = id$ (the identity function), and $h_i(s_i) = e^{k_i s_i}$ ($k_i > 0$), and studies the special case where $k_i = k$ for all i under additional assumptions. In this case, $h_i''(s_i)/h_i'(s_i) = k_i$. So if, in addition, costs are also exponential, $c_i(s_i) = e^{l_i s_i}$, the conclusions of Proposition 1 continue to apply provided that $k_i \leq l_i$.

5.2 Private Provision of Public Goods

We next consider the workhorse model of public good provision originally studied by Bergstrom et al. (1986). There are I individuals, each making a voluntary contribution to the provision of a unique public good. Individual i maximizes her utility function

$$u_i\left(c_i, \sum_{j=1}^{I} s_j + \bar{s}\right)$$

subject to the budget constraint $c_i + ps_i = m_i$. Here $m_i > 0$ is income, c_i private consumption, and s_i is agent i's contribution to the public good, so that $\sum_{j=1}^{I} s_j + \bar{s}$ is total amount of the public good provided. The exogenous variable $\bar{s} \geq 0$ can be thought of as the baseline (pre-existing) level of the public good that will be supplied without any contributions.

Substituting for c_i, it is easily seen that this is an aggregative game with reduced payoff function given by

$$\Pi_i\left(s_i, \sum_{j=1}^{I} s_j, m, p, \bar{s}\right) \equiv u_i\left(m_i - ps_i, \sum_{j=1}^{I} s_j + \bar{s}\right).$$

The aggregator is here simply $g(s) = \sum_{j=1}^{I} s_j$. When $s^* = (s_i^*)_{i\in\mathcal{I}}$ is an equilibrium, we refer to $g(s^*) = \sum_{i=1}^{I} s_i^*$ as the *aggregate equilibrium provision*. Let us simplify the exposition and notation here by assuming that u_i is smooth and that strategy sets are intervals of the type $S_i = [0, \bar{s}_i] \subseteq \mathbb{R}$. The private good will be normal *if and only if* the following condition holds for all $s \in S$:

$$-pD_{12}^2 u_i\left(m_i - ps_i, \sum_{j=1}^{I} s_j + \bar{s}\right) + D_{22}^2 u_i\left(m_i - ps_i, \sum_{j=1}^{I} s_j + \bar{s}\right) \leq 0. \tag{13}$$

for this is that the "share function" approach uses the fact that this function is decreasing everywhere, whereas when $R > 0$, it may be increasing. To see this, apply the implicit function theorem to the condition $\Psi(s_i, Q) = 0$ imposing $h_i(s_i) = s_i$ for all i. Rewrite this in terms of "shares", $z_i = s_i/Q$, so that $[-V_i - (R+Q)^2 c_i'']dz_i = [VRQ^{-2} + (R+Q)^2 c_i'' + c_i' \cdot Q^{-2}(2(R+Q)Q - (R+Q)^2)]dQ$. The coefficient of dz_i is clearly negative. When $R = 0$, the coefficient of dQ on the right-hand-side is unambiguously positive, hence $dz_i/dQ < 0$, *i.e.*, agent i's share function is strictly decreasing. But in general, this may fail when $R > 0$ is allowed. In particular, the term $c_i' \cdot Q^{-2}(2(R+Q)Q - (R+Q)^2)$ will be positive if and only if $Q \geq R$. Clearly, nothing prevents c_i' from being sufficiently large for this term to dominate so that the share function becomes increasing when $Q < R$.

Notice that the left-hand side of (13) is equal to $D^2_{s_i s_j}\Pi_i$. Hence the private good is normal *if and only if* payoff functions exhibit decreasing differences. This then becomes a game with strategic substitutes (cf. Definition 3), and the following result therefore follows directly from the results in Section 3 (proof omitted):

Proposition 2 *Consider the public good provision game and assume that the private good is normal. Then there exists a (pure-strategy) Nash equilibrium. Furthermore:*

1. *An increase in $\bar{s}$ leads to a decrease in the smallest and largest aggregate equilibrium provisions.*

2. *The entry of an additional agent leads to a decrease in the smallest and largest aggregate equilibrium provisions by existing agents.*

3. *A positive shock to agent i will lead to an increase in that agent's smallest and largest equilibrium provisions and to a decrease in the associated aggregate provisions of the remaining $I-1$ players.*

The observation that the public good provision model has a pure strategy Nash equilibrium assuming merely that the private good is normal appears to be new.[28] The absence of any concavity assumptions highlights that results of Proposition 2 could not have been derived using the implicit function theorem.[29]

If instead, we assume that the *public* good is (strictly) normal, we can obtain additional strong results using our findings on nice games from Section 4. Indeed, suppose that the payoff function is pseudo-concave (which was not assumed for Proposition 2). Then the public good will be (strictly) normal *if and only if* the following condition holds for all $s \in S$ (here $Q = g(s)$):[30]

$$D_{s_i}\Psi_i(s_i, Q) = p^2 D_{11}u_i(m_i - ps_i, Q) - pD_{21}u_i(m_i - ps_i, Q) < 0 \tag{14}$$

So the public good is strictly normal if and only if the uniform local solvability condition holds. What is more, (14) implies that an increase in m_i (or a decrease in p) constitutes a positive shock,

[28] Nevertheless, it does follow readily from Novshek (1985) and from Jensen (2010) (see Section 3) once the above connection between normality and strategic substitutes has been made. Note also that convexity of strategy sets and differentiability of payoff functions were assumed here only to simplify the exposition. Proposition 2 is equally valid with, for example, finite strategy sets.

[29] This statement also applies to Corchón (1994), whose comparative statics results on games with strategic substitutes are indeed based on the implicit function theorem. But even ignoring this, it is easy to see that Corchon's "strong concavity assumption" amounts to assuming that both the private and public goods are strictly normal. This "double normality" assumption (as it is often called) dates all the way back to the original article of Bergstrom et al. (1986).

[30] The equivalence between (strict) normality of the public good and (14) follows since $\partial s_i(m, p, \sum_{j\neq i} s_j)/\partial m = \alpha(pD^2_{12}u_i - p^2 D^2_{11}u_i)$.

i.e., $D^2_{s_i m}\Pi_i \geq 0$ and $D^2_{s_i p}\Pi_i \leq 0$, respectively (cf. Definition 9). The next proposition therefore follows immediately from Theorems 6-8 (proof omitted).[31]

Proposition 3 *Consider the public good provision game and assume that the public good is (strictly) normal, that payoff functions are pseudo-concave in own strategies and that strategy sets are convex. Then there exists a (pure-strategy) Nash equilibrium. Furthermore:*

1. *Any positive shock to one or more of the agents (e.g., a decrease in p, or increases in one or more income levels, $m_1, \ldots, m_I$) leads to an increase in the smallest and largest aggregate equilibrium provisions.*
2. *The smallest and largest aggregate equilibrium provisions are increasing in the number of agents.*
3. *The changes in 1 and 2 above are associated with an increase in the provision of agent i if the private good is inferior for this agent, and with a decrease in agent i's provision if the private good is normal and the shock does not directly affect the agent.*

It is also useful to note that Proposition 3 could be obtained under even weaker conditions by using Corollary 3 presented in Section 7 below. In particular, it can be verified that if the public good is normal (condition (14) holding as weak inequality) and payoff functions are quasi-concave (rather than pseudo-concave), the conditions of this corollary are satisfied and Proposition 3 remains valid. We used Theorems 6-8 here since Corollary 3 is not introduced until Section 7.

5.3 The Cournot Model

Considered the Cournot model of quantity competition discussed in Example 2. There are I firms, each choosing $s_i \in [0, \bar{s}_i]$ to maximize profits:

$$\pi_i(s,t) = s_i P\left(\sum_{j=1}^{I} s_j + \bar{t}\right) - c_i(s_i, t_i). \tag{15}$$

Here t_i is a parameter that affects the cost of firm i, and $\bar{t}$ is a parameter affecting inverse demand directly. We assume throughout that $D^2_{s_i t_i} c_i \leq 0$, *i.e.*, that an increase in t_i is a positive shock. Clearly, this is an aggregative with $g(s) = \sum_j s_j$. Moreover, it features strategic substitutes provided that

$$P'(Q + \bar{t}) + s_i P''(Q + \bar{t}) \leq 0, \tag{16}$$

[31] Note in particular that (strict) normality implies local solvability as well as regularity so the statements in Proposition 3 are valid without any boundary conditions on payoff functions.

where $Q \equiv \sum_{j=1}^{I} s_j$.[32] Since this condition does not depend on cost functions, if it holds, an equilibrium will exist regardless of whether profit/payoff functions are concave and/or strategy sets are convex (Novshek (1985), Kukushkin (1994)). Our methods provide general comparative static results for this model.[33] In fact, the following result follows immediately as an application of the theorems provided in Section 3 (proof omitted):

Proposition 4 *Consider the Cournot model and assume that (16) holds. Then this is a game with strategic substitutes and the following comparative statics results apply:*

1. *An increase in $\bar{t}$ leads to a decrease in the smallest and largest aggregate equilibrium outputs.*
2. *The entry of an additional firm leads to a decrease in the smallest and largest equilibrium outputs produced by the existing agents.*
3. *A positive shock to agent i (an increase in t_i) will lead to an increase in that agent's smallest and largest equilibrium outputs and to a decrease in the associated aggregate equilibrium outputs of the remaining $I-1$ firms.*

If instead we were to assume concavity (or pseudo-concavity), comparative statics can be obtained by using results from the existing literature.[34]

5.4 Technology Choice in Oligopoly

As a final application, we consider games in which oligopoly producers make technology choices (as well as setting output). Our treatment here will also illustrate how our results with one-dimensional aggregates can be applied when strategy sets are multidimensional and also clarifies how "perverse" comparative statics may arise in such games and how it can be ruled out. For a general and related discussion of models of technological choice and competition see Vives (2008).

Consider a Cournot model with I heterogeneous firms. Let $q = (q_1, ..., q_I)$ be the output vector and $a = (a_1, ..., a_I)$ the technology vector. Let us define $Q = \sum_{j=1}^{I} q_j$ as aggregate output. Profit

[32] A sufficient condition for (16) is that the elasticity of the P' function $\varepsilon_P(Q) = P''(Q)Q/P'(Q)$ is less than 1(naturally, $P''(Q) \leq 0$ is in turn sufficient for this).

Amir (1996) studies conditions under which the Cournot model will be a game of strategic substitutes or complements (our results on strategic substitutes are equally valid under the ordinal conditions of Milgrom and Shannon (1994) which is what Amir focuses on).

[33] To the best of hour knowledge, there are no existing comparative statics results that apply to the model at this level of generality. To reiterate a point we have already emphasized, this is because the implicit function theorem cannot be used without assuming concavity of payoffs, convexity of strategy sets, interiority, and so forth, and also because in this case results from supermodular games do not apply (except in the special case of two firms).

[34] See however the discussion in footnote 23 of Section 4: In the absence of strategic substitutes (*e.g.*, Amir (1996)), the comparative statics results we get from the theorems in Section 4 are new.

of firm i is

$$\Pi_i (q_i, a_i, Q) \equiv \pi_i (q, a) = q_i P(Q) - c_i (q_i, a_i) - C_i (a_i)$$

where P is the (decreasing) inverse market demand, the cost function c_i is a function of firm i's quantity and technology choices, and C_i is the cost of technology adoption. Assume that P, c_i and C_i (for each i) are twice differentiable, P is strictly decreasing decreasing ($P'(Q) < 0$ for all Q), C_i is convex, and $\partial c_i (q_i, a_i) / \partial q_i \partial a_i < 0$ (for each i), so that greater technology investments reduce the marginal cost of production for each firm.

The first-order necessary conditions for profit maximization are

$$\begin{aligned}
\frac{\partial \pi_i}{\partial q_i} &= P'(Q) q_i + P(Q) - \frac{\partial c_i (q_i, a_i)}{\partial q_i} = 0 \\
\frac{\partial \pi_i}{\partial a_i} &= -\frac{\partial c_i (q_i, a_i)}{\partial a_i} - \frac{\partial C_i (a_i)}{\partial a_i} = 0.
\end{aligned}$$

Naturally, we also require the second-order conditions to be satisfied, which here amount to $D^2_{(q_i, a_i)} \pi_i$ being negative semi-definite. Let us now consider the effect of a decline in the cost of technology investment by one of the firms (*i.e.,* a shift in C_i), which clearly corresponds to a positive shock. The results from Section 4 suggest that we should check the local solvability condition. In particular, consider the matrix

$$D_{(q_i, a_i)} \Psi_i = \begin{pmatrix} P'(Q) - \frac{\partial^2 c_i}{\partial q_i^2} & -\frac{\partial^2 c_i}{\partial q_i \partial a_i} \\ -\frac{\partial^2 c_i}{\partial q_i \partial a_i} & -\frac{\partial^2 c_i}{\partial a_i^2} - \frac{\partial^2 C_i}{\partial a_i^2} \end{pmatrix}$$

for each i. When $c_i (q_i, a_i)$ is convex, the matrix

$$\begin{pmatrix} -\frac{\partial^2 c_i}{\partial q_i^2} & -\frac{\partial^2 c_i}{\partial q_i \partial a_i} \\ -\frac{\partial^2 c_i}{\partial q_i \partial a_i} & -\frac{\partial^2 c_i}{\partial a_i^2} \end{pmatrix}$$

is negative semi-definite. Since $P'(Q) < 0$ and $\partial^2 C_i / \partial a_i^2 \leq 0$, this is sufficient to guarantee that $|D\Psi_i| < 0$. Therefore, whenever each $c_i (q_i, a_i)$ is convex, the local solvability condition is satisfied. Hence, a decline in the cost of technology investments for one of the firms will necessarily increase total output. Similarly, the effects of an increase in demand on output and technology choices can be determined robustly. The following proposition summarizes these results (proof omitted):

Proposition 5 *Consider the technology adoption game described above and assume that the cost functions $c_i = c_i(q_i, a_i)$ (for each i) are convex. Then the local solvability condition holds and as a consequence:*

1. *Any positive shock to one or more of the firms (e.g., a decrease in marginal costs parameterized via $c_i = c_i(q_i, a_i, t)$) will lead to an increase in total equilibrium output.*

2. Entry of an additional firm will lead to an increase in total output.

It is also noteworthy that the oligopoly-technology game is a game with strategic substitutes when $\partial^2 c_i(q_i, a_i)/\partial q_i \partial a_i \leq 0$.[35] So when technological development lowers the marginal cost of producing more input, similar result to those of Proposition 5 follow from our theorems in Section 3. However, notice that without local solvability, the conclusions would be weaker. In the Appendix (Section 10.8), we provide a specific example of the technology adoption game, which exhibits strategic substitutes but violates the local solvability condition, and thus a positive shock may *decrease* (rather than increase) the equilibrium aggregate. That example illustrates that even in "nice" aggregative games with strategic substitutes, the local solvability condition is critical for the conclusion of Theorem 6: unless a shock hits the aggregate (in which case Theorem 3 applies), a positive shock may lead to a *decrease* in the equilibrium aggregate when the local solvability condition does not hold.

6 Multidimensional Aggregates

We have so far focused on aggregative games with one-dimensional aggregates, *i.e.*, games where $g : S \to \mathbb{R}$. Many important examples, require more than a one-dimensional aggregate, $g : S \to \mathbb{R}^M$, $M > 1$. Another game with multidimensional aggregates is the technology choice game considered in Section 5.4 when technology costs also depend on some aggregate of the technology choices of other firms, *e.g.,* $C_i = C_i(a_i, A)$ for some aggregate of technology choices A.

6.1 Theory

We now discuss how our results can be extended to multidimensional aggregates under the additional assumptions. To simplify the exposition we focus on the case where the aggregator takes the form $g(s) = \sum_{j=1}^{I} s_j$.[36] In this case, naturally, $g : S \to \mathbb{R}^N$, hence $M = N$. We continue to assume that there are I players and we denote the set of players by $\mathcal{I}$. In addition, we assume that the game is both "nice" (Definition 6) and also exhibits strategic substitutes. Then, proceeding as in Section 4, we define $D_1\Pi_i(s_i, Q, t) \equiv D_{s_i}\Pi_i(s_i, Q, t)$ and $D_2\Pi_i(s_i, Q, t) \equiv D_Q\Pi_i(s_i, Q, t)$. The marginal payoff for player i can again then be expressed as:

$$D_{s_i}\pi_i(s, t) \equiv D_1\Pi_i\left(s_i, \sum_{j=1}^{I} s_j, t\right) + D_2\Pi_i\left(s_i, \sum_{j=1}^{I} s_j, t\right). \tag{17}$$

[35] This condition ensures that payoff functions are supermodular in own strategies. It is easy to check that payoff functions also exhibit decreasing differences in own and opponents' strategies.

[36] All of the following results remain valid if we assume instead that $g(s) = (g^1(s_1^1, \ldots, s_I^1), \ldots, g^N(s_1^N, \ldots, s_I^N))$ with each function g^n separable. See the beginning of the proof of Theorem 6 for details on how one can transform such a game into a game with a linear aggregator.

Now denoting the vector of aggregates by $Q \equiv \sum_{j=1}^{I} s_j$, we again define:

$$\Psi_i(s_i, Q, t) \equiv D_1\Pi_i(s_i, Q, t) + D_2\Pi_i(s_i, Q, t). \tag{18}$$

Parallel with the local solvability condition (Definition 7 in Section 4), we will place certain key restrictions on the Ψ_i functions. These restrictions, together with our focus on nice games with strategic substitutes, are collected in the following assumption.

Assumption 1 *The game Γ is an aggregative nice game (Definition 6) and in addition, for each player i, we have:*

- ***(Strategic Substitutes)*** *S_i is a compact lattice, and $\pi_i(s_i, s_{-i}, t)$ is supermodular in s_i and exhibits decreasing differences in s_i and s_j (for all $j \neq i$).*
- ***(Strong Local Solvability)*** *Every real eigenvalue of $D_{s_i}\Psi_i\left(s_i, \sum_{j=1}^{I} s_j, t\right)$ is negative.*

Remark 7 ***(Strong Local Solvability)*** That every real eigenvalue of $D_{s_i}\Psi_i\left(s_i, \sum_{j=1}^{I} s_j, t\right)$ is negative implies that its determinant is non-zero (this is because $D_{s_i}\Psi_i$ must have non-negative off-diagonal elements, see the proof of Theorem 10 for further details). Consequently, local solvability (Definition 7) is implied by strong local solvability.

Assumption 1 is straightforward to verify because of the following two relationships linking the usual second-order matrices of π_i and the gradient of the Ψ_i functions:

$$D^2_{s_i s_j}\pi_i(s,t) \equiv D^2_{s_i s_j}\Pi_i\left(s_i, \sum_{k=1}^{I} s_k, t\right) \equiv D_Q\Psi_i\left(s_i, \sum_{k=1}^{I} s_k, t\right) \text{ for all } j \neq i, \text{ and} \tag{19}$$

$$D^2_{s_i s_i}\pi_i(s,t) \equiv D^2_{s_i s_i}\Pi_i\left(s_i, \sum_{j=1}^{I} s_j, t\right) \equiv D_{s_i}\Psi_i\left(s_i, \sum_{j=1}^{I} s_j, t\right) + D_Q\Psi_i\left(s_i, \sum_{j=1}^{I} s_j, t\right). \tag{20}$$

Since by (19), $D_Q\Psi_i \equiv D^2_{s_i s_j}\Pi_i$ for all $j \neq i$, decreasing differences (strategic substitutes) requires simply that $D_Q\Pi_i\left(s_i, \sum_{j=1}^{I} s_j, t\right)$ is a non-positive matrix. Next we can sum the two matrices $D_{s_i}\Psi_i$ and $D_Q\Psi_i$ in order to obtain $D^2_{s_i s_i}\pi_i$ (cf. (19)). Supermodularity holds if and only if the matrix $D^2_{s_i s_i}\Pi_i$ has non-negative off-diagonal entries. Finally, strong local solvability requires that the real eigenvalues of $D_{s_i}\Psi_i$ are negative. When $D_{s_i}\Psi_i$ is symmetric (which is often the case in practice), this is the same as $D_{s_i}\Psi_i$ being a negative definite matrix. Note also that concavity of payoff functions in own strategies is implied by Assumption 1 (see the proof of Theorem 10). Thus, in games with multidimensional aggregates the verification of strong local solvability "replaces" the very similar task of verifying that the Hessian is negative definite. The concavity implications of Assumption 1 also mean that when this assumption holds, the existence of a pure-strategy Nash equilibrium follows immediately by Brouwer's fixed point theorem. This is noted in the following theorem (proof omitted):

Theorem 9 ***(Existence)*** *Suppose that Γ satisfies Assumption 1. Then Γ has a (pure-strategy) Nash equilibrium.*

We next define the backward reply function of player i again using the first-order conditions: $s_i = b_i(Q,t) \Leftrightarrow \Psi_i(s_i, Q) = 0$. Assumption 1 simplifies matters here by ensuring that each Q leads to a *unique* backward reply function (rather than a correspondence), $b_i(Q,t)$.[37] For any given vector of aggregates Q, the gradient of $b_i(Q)$ is also well-defined and is given by:

$$D_Q b_i(Q,t) = -[D_{s_i}\Psi_i(b_i(Q,t),Q,t)]^{-1} D_Q \Psi_i(b_i(Q,t),Q,t), \tag{21}$$

and thus

$$D_Q b(Q,t) = \sum_{j=1}^{I} D_Q b_j(Q,t). \tag{22}$$

Let us also recall that an *M-matrix* is a matrix with positive real eigenvalues and non-positive off-diagonal entries.[38] We are then ready to state the following multidimensional version of Theorems 3 and 6.

Theorem 10 ***(Shocks to the Aggregates)*** *Suppose that Γ satisfies Assumption 1. Let $t \in T \subseteq \mathbb{R}^N$ be a shock to the aggregate, that is, let $\pi_i(s,t) \equiv \Pi_i\left(s_i, t + \sum_{j=1}^{I} s_j\right)$ for all $i \in \mathcal{I}$, and assume that the matrix $\mathbf{I} - [D_Q b(Q+t)]^{-1}$ exists and is non-singular. Then:*

- ***(Sufficiency)*** *If the matrix $\mathbf{I} - [D_Q b(Q+t)]^{-1}$ is an M-matrix (for all Q and t), an increase in $t \in T$ leads to a decrease in each component of the equilibrium aggregate vector.*

- ***(Necessity)*** *Conversely, let $Q(t')$ be an equilibrium aggregate given some vector of parameters $t' \in T$ that hits the aggregate. Then if $\mathbf{I} - [D_Q b(Q(t')+t')]^{-1}$ is not an M-matrix, there exists $t'' > t'$ such that at least one component of the equilibrium aggregate vector increases when t is raised from t' to t''.*

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[37] Fixing Q, it is clear that the gradient of $\Psi_i(\cdot, Q)$, $D_{s_i}\Psi_i(s_i, Q)$ (which is a $N \times N$ matrix), is non-singular at any stationary point. In particular, from strong local solvability, the determinant of $D_{s_i}\Psi_i(s_i, Q)$ never changes sign and never equals zero. This immediately implies that there exists a unique critical point (*e.g.,* from the Poincare-Hopf theorem; Milnor (1965)).

[38] Recall that an M-matrix and an inverse M-matrix are also P-matrices (*i.e.,* all of their principal minors are positive). Moreover, if a matrix has a non-positive off-diagonal, it is an M-matrix if and only if it is also a P-matrix.

Proof. See Section 10.9. ■

In what follows, we will use the sufficiency part of Theorem 10 to present direct parallels to the other theorems presented in Section 3. Nevertheless, the necessity part of this theorem is also noteworthy, perhaps even surprising.

Given Theorem 10, the proofs of the next three theorems closely follow the proofs of the analogous theorems for the one-dimensional case and are thus are omitted.[39] For the next theorem, suppose that the default inaction strategy of the entrant now is a *vector* of zeroes (or, more generally, the least element in the entrant's strategy set).

Theorem 11 ***(Entry)*** *Suppose that Γ satisfies Assumption 1 and the sufficiency conditions in Theorem 10. Then entry of an additional player leads to a decrease in the aggregates of the existing players. In addition, at least one of the aggregates of all players must increase with entry, and strictly so unless the entrant chooses inaction.*

Theorem 12 ***(Idiosyncratic Shocks)*** *Suppose that Γ satisfies Assumption 1 and the sufficiency conditions in Theorem 10. Then a positive idiosyncratic shock to player $i \in \mathcal{I}$ leads to an increase in this player's equilibrium strategy and to a decrease in the associated aggregates of the existing players.*

Remark 8 ***(Sufficient Conditions for Two-Dimensional Aggregates)*** When $N = 2$ ($g : S \to \mathbb{R}^2$), the sufficient conditions are particularly easy to verify. In particular, $\mathbf{I} - [D_Q b(Q+t)]^{-1}$ exists and is a non-singular M-matrix when $-[D_Q b(Q+t)]^{-1}$ is a non-singular M-matrix. This is generally the case (regardless of N), since the real eigenvalues of $[\mathbf{I} - [D_Q b(Q+t)]^{-1}]$ are equal to $(\lambda_1 + 1), \ldots, (\lambda_M + 1)$, where $\lambda_1, \ldots, \lambda_M > 0$ are the real eigenvalues of $-[D_Q b(Q+t)]^{-1}$.

In this two-dimensional case, $-[D_Q b(Q+t)]^{-1}$ is a non-singular M-matrix if and only if the determinant of $D_Q b(Q+t)$ is positive. To see this first note that since $D_Q b(Q+t)$ is a non-positive matrix, its trace is non-positive. So when the determinant is positive, both eigenvalues must be negative (when they are real; if they are not real, then there is nothing to check because the definition of an M-matrix above requires only that the *real* eigenvalues be positive). It then follows that $-[D_Q b(Q+t)]^{-1}$ is a matrix with non-positive off-diagonal elements and positive (real) eigenvalues, and thus it is a non-singular M-matrix.

[39] The only new feature is the second statement of the entry theorem (that at least one of the aggregates must increase upon entry). This is a direct consequence of the fact that the backward reply function of the existing players, $\bar{b}$, is decreasing (this is proved as part of Theorem 10). Indeed, let Q^b be the vector of aggregates *before* entry, Q^a the aggregates *after* entry, and $s_{I+1} \geq 0$ be the strategy chosen by the entrant. Since $\bar{b}$ is decreasing, $Q^a \leq Q^b$ implies: $0 \leq Q^b - Q^a = \bar{b}(Q^b) - \bar{b}(Q^a) - s_{I+1} \leq -s_{I+1}$ which in turn implies $s_{I+1} = Q^b - Q^a = 0$.

Now since $D_Q b(Q+t) = \sum_{j=1}^{I} D_Q b_j(Q+t)$, a sufficient condition for $D_Q b(Q+t)$ to have a positive determinant is that each of the matrices $D_Q b_i(Q+t)$, $i = 1, \ldots, I$ is quasi-negative definite ($x^T D_Q b_i(Q+t) x < 0$ for all $x \neq 0$). This is because the sum of quasi-negative definite matrices is quasi-negative definite, and a 2×2 quasi-negative definite matrix has a positive determinant. The next corollary exploits this observation.

Corollary 2 ***(Symmetric Games with Two-Dimensional Aggregates)*** *Suppose that Γ satisfies Assumption 1 and $N = 2$. Consider a shock to the aggregate. Then if the matrix $D_Q \Psi_i(s_i, \sum_{j=1}^{I} s_j + t)$ has a positive determinant for all $s \in S$ and $t \in T$, a positive shock to the aggregates will lead to a decrease in both of the aggregates in any symmetric equilibrium. In addition, the results in Theorems 11-12 continue to hold when the existing players choose identical strategies before and after entry (Theorem 11), and the players that are not affected by the shock choose identical strategies before and after the arrival of the idiosyncratic shock (Theorem 12).*

Proof. The aggregate in a symmetric equilibrium is given by $Q = I b_i(Q+t)$ where $i \in \mathcal{I}$ is any of the (identical) players. From Theorem 10, a positive shock to the aggregate decreases the aggregate if only if $[\mathbf{I} - [D_Q b(Q+t)]^{-1}$ is a non-singular M-matrix. From Remark 8, we only need to verify that $-[D_Q b(Q+t)]^{-1} = -[D_Q b_i(Q+t)]^{-1}/I$ is a non-singular M-matrix. When $N = 2$, this holds if and only if the determinant of $D_Q b_i(Q+t)$ is positive. This is the case when $D_Q \Psi_i$ has a positive determinant, because $D_{s_i} \Psi_i$ has a positive determinant and $Db_i = -[D_{s_i} \Psi_i]^{-1} D_Q \Psi_i$. ■

6.2 Example

As an example of a game with multidimensional aggregates, consider the oligopoly-technology adoption game discussed in Section 5.4, enriched with an additional payoff interaction from technology choices. More specifically, the profit of firm i is

$$\Pi_i(q_i, a_i, Q) \equiv \pi_i(q, a) = q_i P(Q) - c_i(q_i, a_i) - C_i(a_i, A)$$

where P is the (decreasing) inverse market demand, c_i is the cost of producing output q_i for firm i's as a function of its technology choice, and C_i is the cost of technology adoption, which also depends on aggregate technology decisions summarized by $A = \sum_{j=1}^{I} a_j$ (*e.g.*, through technological spillovers across firms). Clearly, this is an aggregative of game with multidimensional aggregates.

We assume that P, c and C (for each i) are twice differentiable, P is strictly decreasing decreasing ($P'(Q) < 0$ for all Q), all C_is and c_is are convex, and $\partial_i c(q_i, a_i) / \partial q_i \partial a_i < 0$ (for each i), so that greater technology investments reduce the marginal cost of production for each firm.

We also impose additional assumptions that enable us to apply Theorems 10-12 presented in the previous subsection. These are:

1. The condition (16) introduced above is satisfied.

2. We have that for each i,
$$\frac{\partial^2 C_i(a_i, A)}{\partial a_i \partial a_j} + \frac{\partial^2 C_i(a_i, A)}{\partial A^2} \geq 0.$$
This condition implies that there is some degree of "fishing out" in technological choices, in the sense that greater investments by firm j, though typically benefiting firm i, also reduce the marginal effect of firm i's own technology investments on its own cost.

3. We have that for each i,
$$\frac{\frac{\partial^2 c_i(q_i,a_i)}{\partial a_i^2}\frac{\partial^2 c_i(q_i,a_i)}{\partial q_i^2}}{\left(\frac{\partial^2 c_i(q_i,a_i)}{\partial q_i \partial a_i}\right)^2} \geq \frac{\left[\frac{1}{2}(P''(Q)\,q_i + P'(Q) + \left(-\frac{\partial^2 C_i(a_i,A)}{\partial a_i \partial A} - \frac{\partial^2 C_i(a_i,A)}{\partial A^2}\right)\right]^2}{\left(P''(Q)\,q_i + P'(Q)\right)\left(-\frac{\partial^2 C_i(a_i,A)}{\partial a_i \partial A} - \frac{\partial^2 C_i(a_i,A)}{\partial A^2}\right)}.$$
To interpret this condition, note that the left-hand side of the inequality is a measure of convexity of the cost function c_i, whereas the right-hand side is the ratio of the arithmetic to the geometric means of $\partial^2\pi_i(q_i,a_i)/\partial q_i\partial q_j = P''q_i + P'$ and $\partial^2\pi_i(q_i,a_i)/\partial a_i\partial a_j = -\partial^2 C_i/\partial a_i\partial A - \partial^2 C_i/\partial A^2$ (for $i \neq j$). These terms are in turn the "strategic" interaction effects, the first one from quantity choices and the second one from technology choices. This ratio is equal to 1 when the two strategic effects are equal, i.e., $\partial^2\pi_i(q_i,a_i)/\partial q_i\partial q_j = \partial^2\pi_i(q_i,a_i)/\partial a_i\partial a_j$, and is increasing in the degree of "asymmetry of strategic effects". This condition, therefore, requires the cost function to be sufficiently convex to outweigh the asymmetry of strategic effects.

It can be verified that convexity of cost functions ensures both that each firm's problem is supermodular and that the strong local solvability condition is satisfied. Conditions 1 and 2 ensure that this is a game of strategic substitutes. Together with convexity of cost functions, these therefore guarantee that Assumption 1 is satisfied. Condition 3 is a sufficient (though not necessary) condition for the matrix $D_Q b_i(Q+t)$ to be quasi-negative definite as discussed in Remark 8. Consequently, under these conditions the requirements of Theorems 10-12 are met, and strong comparative static results can be obtained in this game. These results are presented in the next proposition and to the best of our knowledge, they could not be derived using just the implicit function theorem or other methods.

Proposition 6 *Consider the technology adoption game with technological spillovers described in this subsection. Assume that the cost functions $c_i = c_i(q_i, a_i)$ and $C_i(a_i, A)$ (for each i) are twice differentiable and convex, and conditions 1-3 above are satisfied. Then:*

1. *A positive demand shock will increase total equilibrium output.*

2. *A positive idiosyncratic shock to one of the firms (e.g., a decrease in marginal costs parameterized via $c_i = c_i(q_i, a_i, t)$) will lead to an increase in this firm's output and technology and reduce the aggregate output and technology of remaining firms.*

3. *Entry of an additional firm will lead to a decrease in the aggregate output and technology of existing firms, and increase either overall aggregate output or aggregate technology or both.*

Proof. See Section 10.10. ■

7 Nice Games without Differentiability

In this section, we extend the results for nice games presented in Section 4. Recall that the main assumption of Section 4, *local solvability*, presupposes that payoff functions and the aggregator are twice continuously differentiable. In this section, we show that robust comparative statics can be derived without differentiability as long as a non-differentiable version of the local solvability condition is imposed. We limit attention to the case of one-dimensional strategy sets (hence the aggregate must be one-dimensional also). Recall that an aggregator always has a representation of the form $g(s) = H\left(\sum_{j=1}^{I} h_j(s_j)\right)$, where H and $h_1, \ldots, h_i$ are strictly increasing functions. Therefore, for any Q in the range of g, we have $Q = g(s) \Leftrightarrow s_i = h_i^{-1}\left[H^{-1}(Q) - \sum_{j\neq i} h_j(s_j)\right]$. Intuitively, this means that if we know the aggregate Q and the strategies of $I-1$ players, we also know the strategy of the last player. Let us also define $G_i(Q, y) \equiv h_i^{-1}[H^{-1}(Q) - y]$. Recall from Milgrom and Shannon (1994) that a function $f(Q, y)$ satisfies the *single-crossing property* in (Q, y) if, for all $Q' > Q$ and $y' > y$, we have

$$f(Q', y) \geq (>)\ f(Q, y) \Rightarrow f(Q', y') \geq (>)\ f(Q, y').$$

The main result in this section, presented next, shows that an appropriately-chosen single-crossing property can replace the local solvability condition (Definition 7) and thus extends our results to nice games without differentiability.

Theorem 13 ***(Comparative Statics for Nice Games without Differentiability)*** *Consider an aggregative game with one-dimensional convex, compact strategy sets, a separable aggregator,* *payoff functions that are upper semi-continuous and quasi-concave in own strategies. Suppose that $\Pi_i(G_i(Q, y), Q, t)$ (for each $i \in \mathcal{I}$) satisfies the single-crossing property in (Q, y). Then the conclusions of Theorems 6 and 7 continue to hold. Moreover, provided that payoff functions are twice differentiable and the equilibrium is interior, the conclusions of Theorem 8 also hold.*

Proof. See Section 10.11. ■

Notice that differentiability is needed in Theorem 8 in order to even state this theorem's main conclusions. Clearly, the more interesting part of Theorem 13 concerns Theorems 6-7.

The next corollary uses the insights of Theorem 13 to provide another useful and simple alternative to the local solvability condition for nice games.

Corollary 3 *Consider a nice aggregative game with linear aggregator $g(s) = \sum_i s_i$, one-dimensional strategy sets (i.e., $N = 1$), and assume that for each player i:*

$$D_{s_i}\Psi_i(s_i, Q) \leq 0 \text{ for all } s_i \text{ and } Q \tag{23}$$

Then the conclusions of Theorems 6, 7, and 8 continue to hold.

Proof. Since g is linear, $G_i(Q, y) = Q - y$ and $\Pi_i(G_i(Q, y), Q, t) = \Pi_i(Q - y, Q, t)$ (for each $i \in \mathcal{I}$). The condition $D_{s_i}\Psi_i(s_i, Q) \leq 0$ is equivalent to $-D_{s_i}\Psi_i = -D^2_{11}\Pi_i - D^2_{21}\Pi_i \geq 0$ for all s_i and Q. This is in turn equivalent to $\Pi_i(Q - y, Q, t)$ exhibiting increasing differences in Q and y. Since increasing differences implies the single-crossing property, the results follow from Theorem 13. ■

Note that Condition (23) in Corollary 3 requires that $D_{s_i}\Psi_i(s_i, Q) \leq 0$ *for all* s_i and Q. By contrast, the local solvability condition requires $D_{s_i}\Psi_i(s_i, Q) \neq 0$, but only when s_i and Q are such that $\Psi_i(s_i, Q) = 0$. Thus neither condition generalizes the other. If (23) holds with *strict* inequality throughout, *i.e.*, $D_{s_i}\Psi_i(s_i, Q) < 0$ for all s_i and Q, then local solvability would be implied, though the weak inequality makes this condition easier to check and apply in a variety of examples (recall the discussion in Section 5.2).[40]

8 Walrasian Play

When the aggregate Q results from the "average" of the strategies of a large number of players, it may plausible to presume that each player $i \in \mathcal{I}$ will *ignore* the effect of its strategy on

[40]It can also be noted that (23) with strict inequality makes up "half" of what Corchón (1994) calls the "strong concavity" condition. The other "half" of Corchon's strong concavity condition requires payoff functions to exhibit strictly decreasing differences in own and opponents' strategies. This is not assumed in our analysis.

the aggregate. In this case, each player $i \in \mathcal{I}$ will maximize the reduced payoff function $\Pi_i = \Pi_i(s_i, Q, t)$ with respect to s_i taking Q as given. This is the behavior assumed in standard general equilibrium theory with a finite (but large) number of households. With analogy, we refer to the situation in aggregative games where players ignore their impact on aggregates as *Walrasian play* and the associated Nash equilibrium as a *Walrasian Nash Equilibrium.*[41] For such games, our results can be strengthened (and the proofs in fact become more straightforward). Here we briefly outline the main results in this case, focusing on one-dimensional aggregates (*i.e.,* $K = 1$).

Definition 10 ***(Walrasian Nash Equilibrium)*** *Consider an aggregative game* $\Gamma = ((\Pi_i, S_i)_{i \in \mathcal{I}}, g, T)$. *The strategy profile* $s^*(t) = (s_1^*(t), \ldots, s_I^*(t))$ *is called a (pure-strategy)* Walrasian Nash equilibrium *given* $t \in T$ *if holding* $Q(t) = g(s^*(t))$ *fixed, we have for each player* $i = 1, \ldots, I$ *that,*

$$s_i^*(t) \in R_i(Q, t) \equiv \arg\max_{s_i \in S_i} \Pi_i(s_i, Q(t), t) \ . \tag{24}$$

Notice that under Walrasian play, a player's best responses $R_i(Q, t)$ will depend on the aggregate Q and the exogenous variables t. An increase in t is a *positive* shock for player i if the smallest and largest selections from $R_i(Q, t)$ are both increasing in t. The game features *strategic substitutes* if each S_i is a lattice, Π_i is supermodular in s_i, and exhibits decreasing differences in s_i and Q. When $N = 1$ and Π_i is twice continuously differentiable, a sufficient condition for t to be a positive shock is that $D^2_{s_i t}\Pi_i(s_i, Q, t) \geq 0$ (for all Q and t), and a sufficient condition for strategic substitutes is that $D^2_{s_i Q}\Pi_i(s_i, Q, t) \leq 0$ (for all Q and t).

As in previous sections, we maintain the compactness and upper semi-continuity assumptions. Also, since as before, there may be multiple equilibria, we continue to focus on the smallest and largest equilibrium aggregates $Q_*(t)$ and $Q^*(t)$ (cf. equations (9) and (8) in Section 3).

Theorem 14 ***(Comparative Statics for Walrasian Nash Equilibria)*** *Consider an aggregative* Γ *and assume that the (reduced) payoff function* $\Pi_i(s_i, Q, t)$ *is quasi-concave in* $s_i \in S_i$ *for each* $i \in \mathcal{I}$. *Then a Walrasian Nash equilibrium exists. Moreover, we have that:*

1. *Theorems 6, 7, and 8 hold for Walrasian Nash equilibria. In particular, a positive shock to one or more of the agents will lead to an increase in the smallest and largest equilibrium aggregates, and entry increases the smallest and largest equilibrium aggregates. In addition, suppose that payoff functions are smooth and the equilibrium is interior. Then for each* $i \in \mathcal{I}$, $s_i^*(t)$ *is locally coordinatewise increasing in a positive shock* t *provided that:*

$$-[D^2_{s_i s_i}\Pi_i(s_i^*(t), Q(t), t)]^{-1} D^2_{s_i Q}\Pi_i(s_i^*(t), Q(t), t) \geq 0,$$

[41] Such "aggregate-taking" behavior has been studied extensively within evolutionary game theory, see for example Vega-Redondo (1997), Possajennikov (2003), and Schipper (2004).

and if $\Pi_i = \Pi_i(s_i, Q)$ *(i.e., the shock does not directly affect player i), then the sign each element of the vector* $D_t s_i^*(t)$ *is equal to the sign of each element of the vector*

$$-[D^2_{s_i s_i}\Pi_i(s_i^*(t), Q(t))]^{-1} D^2_{s_i Q}\Pi_i(s_i^*(t), Q(t)).$$

2. *If* Γ *features strategic substitutes, then Theorems 3, 4, and 5 continue to hold for Walrasian Nash equilibria.*

Proof. (Sketch) For part 1, simply define $Z(Q,t) \equiv \{g(s) \in \mathbb{R} : s_i \in R_i(Q,t) \text{ for all } i\}$. In view of our compactness and upper semi-continuity assumptions, $Z(Q,t)$ is a convex-valued, upper hemi-continuous correspondence. Then we can proceed as in the proof of Theorem 6 but using $Z(Q,t)$ instead of the function $\overline{q}$. Figure 8 in the Appendix makes it clear that the general argument remains valid if instead of the function $\overline{q}$, we use a convex-valued correspondence. Given this result, the proofs of Theorems 7, and 8 apply with minimal modifications.

For part 2, note that a shock to the aggregate is a negative shock (by decreasing differences in s_i and Q), hence it leads to a decrease in the smallest and largest aggregates by the conclusion from part 1. The conclusions of Theorems 4 and 5 are established by straightforward modifications of the original proofs. ■

A noteworthy implication of Theorem 14 is that all our results for nice aggregate games continue hold for a Walrasian Nash equilibria without imposing local solvability or differentiability and boundary conditions (only quasi-concavity is imposed to ensure that best response correspondences are convex-valued). This highlights that the challenge in deriving robust comparative static results in aggregative games lies in limiting the magnitude of the effect of own strategies on the aggregate. It should also be noted that part *2* of the theorem is false if payoff functions are not assumed to be quasi-concave, though the results do hold if the game instead features *strict strategic substitutes* (*i.e.*, if strictly decreasing differences is assumed instead of decreasing differences in Definition 3).[42] The proof of Theorem 14 also shows that any separability assumptions on the aggregator g are unnecessary: the conclusions hold provided that g is an increasing function (without further restrictions).

9 Conclusion

This paper presented robust comparative static results for aggregative games and showed how these results can be applied in several diverse settings. In aggregative games, each player's payoff

[42] However, an equilibrium is not guaranteed to exist in this case because of lack of quasi-concavity. To apply the result mentioned in the text one must thus first (directly) establish the existence of an equilibrium.

depends on her own actions and on an aggregate of the actions of all players (for example, sum, product or some moment of the distribution of actions). Many common games in industrial organization, political economy, public economics, and macroeconomics can be cast as aggregative games. Our results focused on the effects of changes in various parameters on the aggregates of the game. In most of these situations the behavior of the aggregate is of interest both directly and also indirectly, because the comparative statics of the actions of each player can be obtained as a function of the aggregate. For example, in the context of a Cournot model, our results characterize the behavior of aggregate output, and given the response of the aggregate to a shock, one can then characterize the response of the output of each firm in the industry.

We focused on two classes of aggregative games: (1) aggregative of games with strategic substitutes and (2) "nice" aggregative games, where payoff functions are twice continuously differentiable, and (pseudo-)concave in own strategies. For example, for aggregative games with strategic substitutes, we showed that:

1. Changes in parameters that only affect the aggregate always lead to an increase in the aggregate (in the sense that the smallest and the largest elements of the set of equilibrium aggregates increase).

2. Entry of an additional player decreases the (appropriately-defined) aggregate of the strategies of existing players.

3. A "positive" idiosyncratic shock, defined as a parameter change that increases the marginal payoff of a single player, leads to an increase in that player's strategy and a decrease in the aggregate of other players' strategies.

We provided parallel, and somewhat stronger, results for nice games under a local solvability condition (and showed that such results do not necessarily apply without this local solvability condition).

The framework developed in this paper can be applied to a variety of settings to obtain "robust" comparative static results that hold without specific parametric assumptions. In such applications, our approach often allows considerable strengthening of existing results and also clarifies the role of various assumptions used in previous analysis. We illustrated how these results can be applied and yield sharp results using several examples, including public good provision games, contests, and oligopoly games with technology choice.

Our results on games with multidimensional aggregates (Section 6) are only a first step in this direction and our approach in this paper can be used to obtain additional characterization results for such games. We leave a more systematic study of games with multidimensional aggregates to future work. We also conjecture that the results presented in this paper can be generalized to games with infinitely many players and with infinite-dimensional strategy sets. In particular, with the appropriate definition of a general aggregator for a game with infinitely many players (*e.g.,* along the lines of the separability definitions in Vind and Grodal (2003), Ch. 12-13), our main results and in fact even our proofs remain valid in this case. Similarly, with the appropriate local solvability condition in infinite dimension, all of our results also appear to generalize to games with infinite-dimensional strategy sets. The extension of these results to infinite-dimensional games is another area for future work.

10 Appendix

10.1 An Example of "Perverse" Comparative Statics

Consider three players $i = 1, 2, 3$ with payoff functions $\pi_i(s) = -0.5s_i^2 + \alpha_i(1-\alpha_i)^{-1}(\sum_{j\neq i} s_j)s_i + \beta_i(1-\alpha_i)^{-1}s_i$ defined locally in a sufficiently large neighborhood of the equilibrium found below. Assume that $\alpha_1 > 1$, $1 > \alpha_2 > 0$, $\alpha_3 < 0$, $\beta_1 < 0$, $\beta_2 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 > 1$, $\beta_1 + \beta_2 + \beta_3 < 0$, and $\alpha_1 + \alpha_3 < 1$.

This is an aggregative game since we can write the payoffs as a function of players' own strategies and the aggregate $Q = \sum_j s_j$: $\Pi_i(s_i, Q) = -0.5s_i^2 + \alpha_i(1-\alpha_i)^{-1}(Q-s_i)s_i + \beta_i(1-\alpha_i)^{-1}s_i$. By strict concavity, best response functions in this game are: $r_i(s_{-i}) = \alpha_i(1-\alpha_i)^{-1}(\sum_{j\neq i} s_j) + \beta_i(1-\alpha_i)^{-1}$. Solving for the pure strategy Nash equilibrium ($s_i^* = r_i(s_{-i}^*)$, $i = 1, 2, 3$) we find a unique equilibrium given by: $s_i^* = \alpha_i Q^* + \beta_i$, where $Q^* = s_1^* + s_2^* + s_3^* = (\beta_1 + \beta_2 + \beta_3)^{-1}(1 - \alpha_1 - \alpha_2 - \alpha_3)$ is the equilibrium aggregate. Now consider a (small) increase in α_2. This is a "positive shock" to player 2: holding opponents' strategies fixed, it increases player 2's marginal payoff and therefore "lifts" player 2's best response function, $\partial r_2(s_{-2})/\partial \alpha_2 > 0$.[43] But this positive direct effect on player 2's optimal strategy notwithstanding, an increase in α_2 leads to a decrease in player 2's strategy in equilibrium:

$$\frac{\partial s_i^*}{\partial \alpha_2} = Q^* + \alpha_2 \frac{\partial Q^*}{\partial \alpha_2} = \frac{\beta_1 + \beta_2 + \beta_3}{1 - \alpha_1 - \alpha_2 - \alpha_3} + \alpha_2 \frac{\beta_1 + \beta_2 + \beta_3}{(1 - \alpha_1 - \alpha_2 - \alpha_3)^2} < 0$$

As can also be seen, the positive shock to player 2 leads to a decrease in the equilibrium aggregate:

$$\frac{\partial Q^*}{\partial \alpha_2} < 0.$$

In summary, a parameter change that unambiguously increases the marginal payoff for a player, which should, all else equal, lead to an increase in that player's strategy and the aggregate, in fact leads to a decrease in the player's strategy in equilibrium as well as a decrease in the aggregate. This happens even though payoff functions are smooth and strictly concave, and the equilibrium is unique, interior, and varies continuously with the exogenous variable α_2.

10.2 Proof of Theorem 2

For each player i, define the correspondence $\mathrm{Gr}[R_i] : T \to 2^S$ by,

$$\mathrm{Gr}[R_i](t) \equiv \{s \in S : s_i \in R_i(s_{-i}, t)\} \ , t \in T$$

This correspondence is upper hemi-continuous and has a closed graph: if $s_i^m \in R_i(s_{-i}^m, t^m)$ for a convergent sequence $(s^m, t^m) \to (s, t)$, then by the fact that R_i itself has a closed graph, $s_i \in R_i(s_{-i}, t)$. Moreover, $E(t) = \cap_i \mathrm{Gr}[R_i^t]$. The correspondence $E : T \to 2^S$ is thus given by the intersection of a finite number of upper hemi-continuous correspondences, and so is itself upper hemi-continuous. In particular, E has compact values ($E(t) \subseteq S$, where S is compact). Therefore, the existence of the smallest and largest equilibrium aggregates, $Q_*(t)$ and $Q^*(t)$, follows from the continuity of g and from Weierstrass' theorem. Upper semi-continuity of $Q^* : T \to \mathbb{R}$ follows

[43] In other words, player 2's payoff function exhibits *increasing differences* in s_2 and α_2 (Topkis (1978)). This is an equivalent way of defining a "positive shock" when strategy sets are one-dimensional and payoff functions are concave.

directly from the fact that g is upper semi-continuous and E is upper hemi-continuous (see Ausubel and Deneckere (1993), Theorem 1). Lower semi-continuity of Q_* follows by the same argument since $Q_*(t) \equiv -\max_{s\in E(t)} -g(s)$ and g is also lower semi-continuous. Finally, when the equilibrium aggregate is unique for all t, $Q_*(t) = Q^*(t)$ and so is both upper and lower semi-continuous and thus continuous in t on T. ■

10.3 Proof of Theorem 3

Recall that $\pi_i(s,t) \equiv \Pi_i(s_i, G(g(s), t))$ all i, where $g : S \to \mathbb{R}$ is separable in s and $G(g(s), t)$ is separable in (s,t). This implies that,

$$G(g(s), t) = H\left(h_T(t) + \sum_{j=1}^{I} h_j(s_j)\right),$$

where $g(s) = M\left(\sum_{i\in\mathcal{I}} h_i(s_i)\right)$ is the aggregator of Definition 1, and M is a strictly increasing function. Moreover, recall that the best response correspondence $R_i(s_{-i}, t)$ is upper hemi-continuous for each $i \in \mathcal{I}$. Let $h_i(S_i)$ be the image of the strategy set S_i under $h_i(\cdot)$ and define the "reduced" best response correspondence $\tilde{R}_i\left(h_T(t) + \sum_{j\neq i} h_j(s_j)\right) \equiv R_i(s_{-i}, t)$ for each i. We can then define the following upper hemi-continuous (possibly empty-valued) correspondence for each player i:

$$B_i(Q,t) \equiv \{\eta \in h_i(S_i) : \eta \in h_i \circ R_i(h_T(t) + Q - \eta)\} \tag{25}$$

Let

$$Z(Q,t) \equiv \sum_{j=1}^{I} B_j(Q,t)$$

be the *aggregate backward reply correspondence* associated with the aggregate $Q = \sum_j h_j(s_j)$. Clearly, the "true" aggregate $g(s) = M\left(\sum_j h_j(s_j)\right)$ is monotonically increasing in the aggregate $Q = \sum_j h_j(s_j)$. Therefore, we may without loss of generality focus on $\sum_j h_j(s_j)$ instead of $g(s)$ in the following. We shall sometimes go further and refer to $Q = \sum_j h_j(s_j)$ as the equilibrium aggregate in order to simplify the exposition.

Let $q(Q,t) \in Z(Q,t)$ be the "Novshek-selection" shown as the thick segments in the figure below. The precise definition of this selection follows next.[44]

Definition 11 ***(Novshek Selections)*** *Let $Q^a, Q^b \in \mathbb{R}$, $Q^a \leq Q^b$. A selection $q : [Q^a, Q^b] \to \mathbb{R}$ from Z (i.e., a function with $q(Q,t) \in Z(Q,t)$ for all $Q \in [Q^a, Q^b]$) is called a Novshek selection (on $[Q^a, Q^b]$) if the following hold for all $Q \in [Q^a, Q^b]$:*

1. *$q(Q,t) \geq z$ for all $z \in Z(Q,t)$.*
2. *$q(Q,t) \leq Q$.*
3. *The backward reply selections $b_i(Q,t) \in B_i(Q,t)$ associated with q (i.e., backward reply selections satisfying $q(Q,t) = \sum_j b_j(Q,t)$ all Q) are all decreasing in Q on $[Q^a, Q^b]$,* i.e., *$Q'' > Q' \Rightarrow b_i(Q'',t) \leq b_i(Q',t)$.*

[44] The construction here is slightly different from the original one in Novshek (1985), but the basic intuition is the same. Aside from being somewhat briefer, the present way of constructing the "Novshek-selection" does not suffer from the "countability problem" in Novshek's proof pointed out by Kukushkin (1994), since we use Zorn's Lemma to construct the selection.

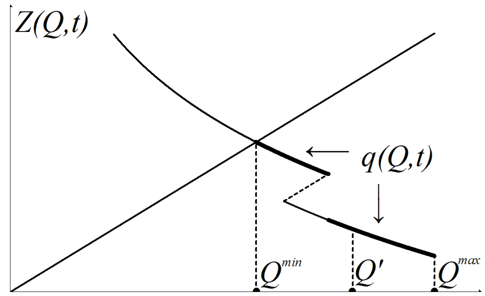

Figure 1: Constructing the aggregate "Novshek-selection"

Before we can construct a suitable Novshek selection, we need to establish the existence of an element $Q^{max} > 0$ as in the figure, with the property that $q < Q^{max}$ for all $q \in Z(Q^{max}, t)$. This can be done by suitably modifying an argument of Kukushkin (1994) (p. 24, l.18-20).

Lemma 1 *There exists an element $Q^{max} > 0$ such that $q < Q^{max}$ for all $q \in Z(Q^{max}, t)$.*

Proof. Let D_i denote the subset of $\mathbb{R}$ upon which $h_i \circ \tilde{R}_i$ is defined, *i.e.*, write $\gamma \in D_i$ if and only if $h_i \circ \tilde{R}_i(\gamma) \neq \emptyset$. Since $h_i \circ \tilde{R}_i$ is upper hemi-continuous, D_i is closed. It is also a bounded set since $\tilde{R}_i \subseteq S_i$ and each S_i is compact. Consequently, D_i has a maximum, which we denoted by d_i. Then extend $h_i \circ \tilde{R}_i$ from D_i to $D_i \cup (d_i, Q^{max}]$ by taking $h_i \circ \tilde{R}_i(d) \equiv \perp_i$ all $d \in (d_i, Q^{max}]$. Here $\perp_i$ can be any small enough element (for each player $i \in \mathcal{I}$) such that $\sum_i \perp_i < Q^{max}$, $\perp_i \leq \min h_i \circ \tilde{R}_i(d_i)$, and $Q^{max} - \perp_i \in (d_i, Q^{max}]$. Defining the aggregate backward reply correspondence Z as above, it is clear that $Z(Q^{max}, t) = \{\sum_i \perp_i\} < Q^{max}$. ∎

Let $D \subseteq (-\infty, Q^{max}]$ denote the subset of $\mathbb{R}$ upon which $Z(\cdot, t)$ is well-defined, *i.e.*, the set of those $Q \leq Q^{max}$ for which $B_i(Q, t) \neq \emptyset$ for all i. Abusing notation slightly, let $[Q', Q^{max}] \equiv D \cap \{Q : Q' \leq Q \leq Q^{max}\}$. Any such interval $[Q', Q^{max}]$ will be compact because D is compact (see the proof of the previous lemma for an identical argument).

Lemma 2 *There exists an element $Q^{min} \leq Q^{max}$ and a well-defined Novshek selection $q : [Q^{min}, Q^{max}] \to \mathbb{R}$ on $[Q^{min}, Q^{max}]$. The element Q^{min} will be minimal in the sense that if $Q' < Q^{\min}$, then there will not exist a Novshek selection on $[Q', Q^{\max}]$.*

Proof. Denote by $\Omega \subseteq 2^{\mathbb{R}}$ the *set* of all "intervals" $[Q', Q^{max}]$ upon which a selection with properties 1.-3. exists. Notice that $\{Q^{max}\} \in \Omega$ so Ω is not empty. Ω is ordered by inclusion since for any two elements ω', ω'' in Ω, $\omega'' = [Q'', Q^{max}] \subseteq [Q', Q^{max}] = \omega' \Leftrightarrow Q'' \leq Q'$. A *chain* in Ω is a totally ordered subset (under inclusion). It follows directly from the upper hemi-continuity of the backward reply correspondences that any such chain with an upper bound has a supremum in Ω (*i.e.*, Ω contains an "interval" that contains each "interval" in the chain). Zorn's Lemma therefore implies that Ω contains a maximal element, *i.e.*, there exists an interval $[Q^{min}, Q^{max}] \in \Omega$ that is not (properly) contained in any other interval from Ω. ∎

Given the Novshek selection of the previous lemma, existence of an equilibrium aggregate (hence a PSNE) is established by showing that $q(Q^{min}) = Q^{min}$ (it is clear that any Q with $q(Q) = Q$ will be an equilibrium aggregate). A sketch of this proof follows next. One easily sees that the equilibrium aggregate Q^{min} thus determined is the *largest* equilibrium aggregate (since if $Q' \in Z(Q', t)$ for $Q' > Q^{min}$, the Novshek selection would have to have $q(Q') = Q'$, or else condition *1.* of Definition 11 would be violated).

Lemma 3 $q(Q^{min}) = Q^{min}$.

Proof. The main step in proving the existence of equilibrium consists in showing that Q^{min} is an equilibrium aggregate (it is easy to see that if Q^{min} is an equilibrium aggregate, then the associated backward replies form an equilibrium). Since we have $q(Q^{min}, t) \leq Q^{min}$ by construction, this can be proved by showing that $q(Q^{min}, t) < Q^{min}$ cannot happen. This step is completed by showing that if $q(Q^{min}, t) < Q^{min}$ holds, then q can be further extended "to the left" (and the extension will satisfy (i)-(iii) above). This would violate the conclusion of Zorn's Lemma that $[Q^{min}, Q^{max}]$ is maximal, thus leading to a contradiction. The details of this step are identical to those in Novshek (1985) and are omitted. ■

We are now ready to prove the main claim of the theorem, namely that the largest equilibrium aggregate Q^{min} characterized above, will be decreasing in t (for the case of the smallest equilibrium aggregate, see the end of this subsection). The proof is non-trivial, and to make it more accessible we shall illustrate it graphically followed by precise formal arguments in all cases. We note that it is sufficient to establish this result for all local changes in t since if a function is decreasing at all points, it is globally decreasing (of course, the associated equilibrium aggregate may well jump - the argument is local only in regard to changes in t).

First, note that since h_T is an increasing and continuous function, any selection $\hat{q}(Q, t)$ from Z will be locally decreasing in Q if and only if it is locally decreasing in t (this follows immediately from the definition of B_i in (25)). Likewise, such a selection $\hat{q}(Q, t)$ will be locally continuous in Q if and only if it is locally continuous in t. Figures 2-5 illustrate the situation for $t' < t''$. The fact that the direction of the effect of a change in Q and t is the same accounts for the arrows drawn. In particular, any increasing segment on the graph of Z will be shifted up when t is increased, and any decreasing segment will be shifted down.

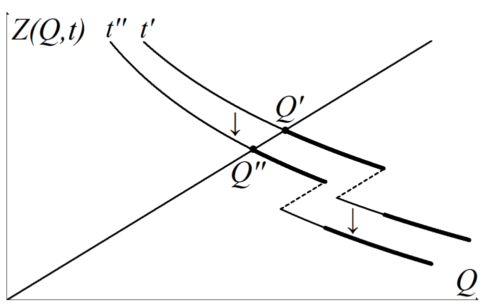

Figure 2: Case I

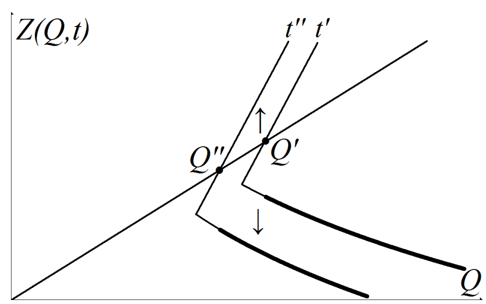

Figure 3: Case II

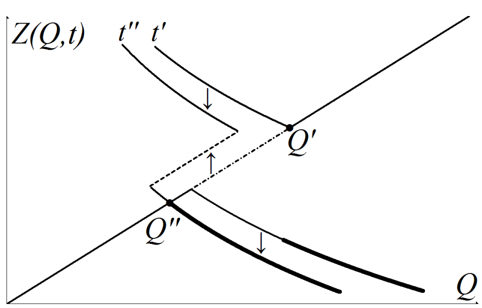

Figure 4: Case III

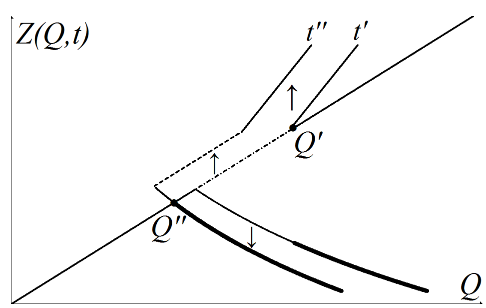

Figure 5: Case IV

There are four cases: Either the graph of Z's restriction to a neighborhood of Q^{min} is locally continuous in Q (equivalently, t), and this function is decreasing in Q and t (Case I) or increasing in Q and t (Case II).[45] Otherwise, continuity does not obtain which is the same as saying that the equilibrium aggregate must "jump" when t is either increased from t' to t'' (Cases III and IV) or decreased from t'' to t' [If t is decreased, case III reduces to Case I and Case IV reduces to Case II.]

Cases III and IV are easily dealt with: If the equilibrium aggregate jumps, it necessarily jumps *down* (and so is decreasing in t). The reason is that an increase in t will always correspond to the graph of Z (locally) being shifted to "the left"(that is another way of saying that any increasing segment will be shifted up, and any decreasing segment shifted down which was the formulation used above). Hence no new equilibrium above the original largest one can appear, the jump has to be to a lower equilibrium (this is also immediate in light of the figures). We now consider the more difficult Cases I and II in turn. Throughout $\hat{q}$ denotes the graph of Z's restriction to a neighborhood of $Q^{\min}$, and Q' and Q'' refer to the (largest) equilibrium aggregates associated with t' and t'', respectively.

Case I: In this case we have $\underline{Q} < \overline{Q}$ such that $\hat{q}(\underline{Q}, t) - \underline{Q} > 0$ and $\hat{q}(\overline{Q}, t) - \overline{Q} < 0$, and such that the new equilibrium aggregate Q'' lies in the interval $[\underline{Q}, \overline{Q}]$. Since $\hat{q}$ is decreasing in t, it immediately follows that $Q'' \leq Q'$. This is what we wanted to show. Note that this observation actually does not depend on continuity of $\hat{q}$ in Q, but merely on the fact that a new equilibrium aggregate Q'' exists and lies in a neighborhood of Q' in which $\hat{q}$ is decreasing (in other words, given that $\hat{q}$ is decreasing, it depends solely on the fact that the aggregate does *not* "jump").

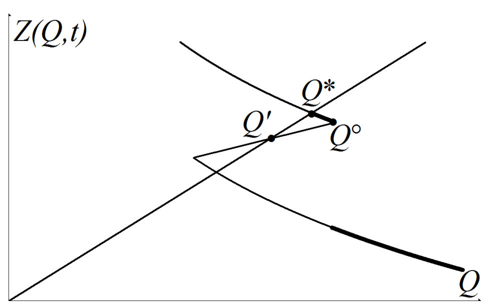

Figure 6: Slope below 1 is impossible: Q' being largest equilibrium aggregate violates that $q(Q,t)$ is decreasing in Q.

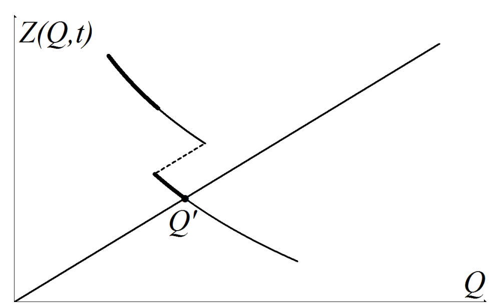

Figure 7: The "Novshek selection" leading to the smallest equilibrium aggregate.

Case II: When $\hat{q}$ is (locally) increasing, we *must* have $\underline{Q} < Q' < \overline{Q}$ such that $\underline{Q} - \hat{q}(\underline{Q}, t) > 0$ and $\overline{Q} - \hat{q}(\overline{Q}, t) < 0$. Intuitively, this means that the slope of $\hat{q}$ is greater than 1 at the point Q' as illustrated in Figure 3. Formally, this can be proved as follows: Assume that there exists $Q^\circ > Q'$ such that $Q^\circ - \hat{q}(Q^\circ, t) \leq 0$ (intuitively this means that the slope is below unity, see Figure 6). Then since $\hat{q}(Q^\circ, t) \geq Q^\circ > Q'$, no Novshek selection could have reached Q' and there would consequently have to be a larger equilibrium Q^*, which is a contradiction.

We now prove that the equilibrium aggregate is decreasing in t: $Q'' \leq Q'$. As in the previous case, we prove this without explicit use of continuity (the proof is straightforward if continuity is used directly as seen in Figure 5). In particular, let us establish the stronger statement that $C(t) \equiv h_T(t) + Q(t)$ is decreasing in t where $Q(t)$ is the largest equilibrium aggregate given t (since $h_T(t)$ is increasing in t, it is obvious that $Q(t)$ must be decreasing in t if $C(t)$ is decreasing). Define the following function: $f(C,t) = C - h_T(t) - \hat{q}(C - h_T(t), t)$. Clearly $C(t) = h_T(t) + Q(t)$ as defined with $Q(t)$ an equilibrium if and only if $f(C(t), t) = 0$. Let $\underline{C} = h_T(t) + \underline{Q}$ and $\overline{C} = h_T(t) + \overline{Q}$. Now

[45] Of course q will in these and all other cases be decreasing where it is defined as the figures also show.

what was proved in the previous paragraph comes into play since it allows us to conclude that: $f(\underline{C},t) = \underline{Q} - \hat{q}(\underline{Q},t) > 0$ and $f(\overline{C},t) = \overline{Q} - \hat{q}(\overline{Q},t) < 0$. Since $B_i(C - h_T(t),t)$ is independent of t (t cancels out in the definition of the backward reply correspondence), $\hat{q}(C - h_T(t),t)$ must be constant in t, *i.e.*, $\hat{q}(C - h_T(t),t) = \tilde{q}(C)$ for some function $\tilde{q}$ which is increasing (since we are in Case II). So f can be written as $f(C,t) = C - h_T(t) - \tilde{q}(C)$ where $\tilde{q}$ is increasing, and consequently f will be decreasing in t and Q. Considering the solution to $f(C,t) = 0$ given t, *i.e.*, $C(t)$, it immediately follows that if t increases then $C(t)$ must decrease. This finishes the proof of the claim in Case IV.

Remark 9 The fact that the term $h_T(t) + Q(t)$ is decreasing in t implies that, in this case, when there is entry of an additional player, the aggregate of all of the agents (including the entrant) decreases. To see this, compare with the proof of Theorem 5 and use that the aggregate of interest is $\sum_j h_j(s_j) + h_{I+1}(s_{I+1})$ (in other words, take $h_T(t) = h_{I+1}(s_{I+1})$).

Combining the observations made so far shows that the largest equilibrium aggregate is decreasing in t as claimed in the theorem. None of the previous conclusions depend on continuity of q in Q, and it is straightforward to verify that the same conclusions hold regardless of whether Q lies in a convex interval (strategy sets could be discrete, say).[46] The statement for the smallest equilibrium aggregate can be shown by an analogous argument. In particular, instead of considering the selection $q(Q,t)$ one begins with Q sufficiently low and studies the backward reply correspondence above the 45° line, now choosing for every Q the smallest best response (Figure 7). This completes the proof of Theorem 3. ■

10.4 Proof of Theorem 5

Let $\tilde{R}_i$ denote the "reduced" backward reply correspondence defined by $\tilde{R}_i\left(\sum_{j\neq i} h_j(s_j), t\right) \equiv R_i(s_{-i},t)$ for each i. To simplify notation, let us set $i = 1$ (assume that the idiosyncratic shock hits the first player, in particular then $\tilde{R}_i$ is independent of $t = t_1$ for all $i \neq 1$). Any pure-strategy Nash equilibrium will also be a fixed point of the set-valued equilibrium problem: $s_1 \in \tilde{R}_1\left(\sum_{j\neq 1} h_j(s_j), t_1\right)$ and $h_i(s_i) \in h_i \circ \tilde{R}_i\left(\sum_{j\neq i} h(s_j)\right)$ for $i = 2,\ldots,I$. Consider the last $I-1$ inclusions, rewritten as

$$h_i(s_i) \in h_i \circ \tilde{R}_i\left(\left(\sum_{j\neq i,1} h_j(s_j)\right) + h_1(s_1)\right) \text{ for } i = 2,\ldots,I. \tag{26}$$

For given $s_1 \in S_1$, Theorem 3 implies that there exist a smallest and largest scalars $y_*(s_1)$ and $y^*(s_1)$ and solutions to the $I-1$ inclusions in (26) which $y_*(s_1) = \sum_{j\neq 1} h_j(s_{j,*})$ and $y^*(s_1) = \sum_{j\neq 1} h_j(s_j^*)$, respectively. In addition, y_*, $y^* : S_1 \to \mathbb{R}$ are decreasing functions.

Now combining y_* and y^* that solve (26) in the sense described, with $s_1 \in \tilde{R}_1\left(\sum_{j\neq 1} h_j(s_j), t_1\right)$ and replacing s_1 with $\bar{s}_1 = -s_1$, we obtain a system with two inclusions:

$$\bar{s}_1 \in -\tilde{R}_1(y, t_1)$$

and

$$y \in \{y_*(-\bar{s}_1), y^*(-\bar{s}_1)\}.$$

This system is ascending in $(\bar{s}_1, y)$ in the sense of Topkis (1998), hence its smallest and largest fixed points are decreasing in t_1 (since the system is descending in t_1 in the sense of Topkis). Therefore,

[46] See Kukushkin (1994) for the details of how the backward reply selection is constructed in such non-convex cases.

the smallest and largest equilibrium strategies for player 1 are increasing in t_1, while the associated aggregate of the remaining players y is decreasing in t. That the smallest and largest strategies for player 1 do in fact correspond to the smallest and largest strategies in the original game is straightforward to verify: Clearly, $y_*(s_1)$ and $y^*(s_1)$ are the smallest and largest aggregates of the remaining players (across all strategy profiles compatible with an equilibrium given s_1), and since $\tilde{R}_1$ is descending in y, the corresponding equilibrium values of s_1 are, respectively, the largest and the smallest.

Finally, it follows by construction that the corresponding aggregates of the remaining players must be the smallest and largest for the original game. This completes the proof of the theorem. ■

10.5 Proof of Theorem 6

We begin by noting that there is no loss of generality in using the aggregator $g(s) \equiv \sum_i h_i(s_i)$ in the following, and assuming that $\min_{s_i \in S_i} h_i(s_i) = 0$ for all i. To see why, recall that the local solvability condition is independent of any strictly increasing transformation of the aggregator as well as any coordinate shift (Remark 6). Let the original aggregator be $\tilde{g}(s) = H(\sum_i \tilde{h}_i(s_i))$. We begin by transforming strategy vectors by multiplying with a positive constant b_i such that $\max_{s_i \in S_i} \tilde{h}_i(s_i) - \min_{s_i \in S_i} \tilde{h}_i(s_i) = 1$. Next, we use the transformation $f(z) = H^{-1}(z) - \sum_i \min_{s_i \in S_i} \tilde{h}_i(s_i)$ to get the new aggregator $g(s) \equiv f(\tilde{g}(s)) = \sum_i h_i(s_i)$, where $h_i(s_i) \equiv \tilde{h}_i(s_i) - \min_{s_i \in S_i} \tilde{h}_i(s_i)$. Clearly, $\min_{s_i \in S_i} h_i(s_i) = 0$ all i with this transformed aggregator.

Let $R_i : S_{-i} \times T \to S_i$ be the best response correspondence of player i and $\tilde{R}_i$ the transformed and reduced best response correspondence defined by $\tilde{R}_i(\sum_{j \neq i} h_j(s_j), t) \equiv h_i \circ R_i(s_{-i}, t)$. Then define the (transformed) *backward reply correspondence* B_i of player i by means of:

$$\eta_i \in B_i(Q, t) \Leftrightarrow \eta_i \in \tilde{R}_i(Q - \eta_i, t)$$

It is clear that Q is an equilibrium aggregate given $t \in T$ if and only if $Q \in Z(Q, t) \equiv \sum_i B_i(Q, t)$ (the correspondence Z is the aggregate backward reply correspondence already studied in the proof of Theorem 3).

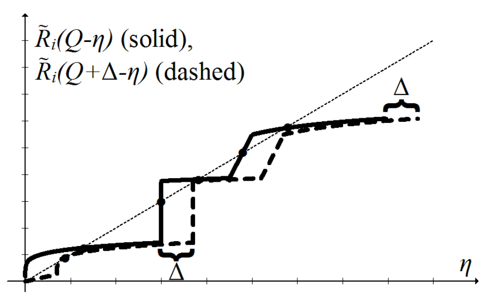

Figure 8: $\eta \in B_i(Q)$ [$\eta \in B_i(Q + \Delta)$] if and only if the solid [dashed] curve intersects the diagonal at η.

We are going to suppress t to simplify notation in what follows. By definition, $\eta \in B_i(Q) \Leftrightarrow \eta \in \tilde{R}_i(Q - \eta)$. Graphically, η lies in B_i if and only if the correspondence $\tilde{R}_i(Q - \cdot)$ intersects with the diagonal/45°-line at η. A crucial feature of the graphs of $\tilde{R}_i(Q - \cdot)$ for different values of Q, is that these correspond to "horizontal parallel shifts" of each other. To be precise, consider the solid curve in figure 8 which is the graph of $\tilde{R}_i(Q - \cdot)$ for some choice of Q. Now increase Q to $Q + \Delta$, $\Delta > 0$. Because of the *additive* way in which η and Q enter into $\tilde{R}_i$, the graph of $\tilde{R}_i(Q + \Delta - \cdot)$ will precisely be a *parallel right shift* of the graph of $\tilde{R}_i(Q - \cdot)$ with each point on the former laying precisely Δ to the right of each point on the latter (the dotted curve in figure 8). Similarly, if $\Delta < 0$, the graph will be shifted to the *left* in a parallel fashion. It is this straight-forward observation that drives essentially the entire proof together with the following: In the case where S_i is of dimension greater than 1, we have by our assumptions that $\eta_i \in B_i(Q,t)$ if and only if $\Psi_i(s_i, Q) = 0$ for some $s_i \in S_i$ with $\eta_i = h_i(s_i)$. When $S_i \subseteq \mathbb{R}$, a similar statement is valid for any η_i in the interior of S_i: $\eta_i \in B_i(Q,t) \Leftrightarrow \Psi_i(\eta_i, Q) = 0$. Crucially, even if we are in the one-dimensional case where it is possible that $\eta_i \in B_i(Q)$ is on the boundary of S_i without the first-order conditions holding, we may because of the uniform local solvability condition assume without loosing generality that $\eta_i \in B_i(Q) \Leftrightarrow \Psi_i(\eta_i, Q) = 0$. The verification of this claim is somewhat technical and is placed in a footnote.[47]

Lemma 4 *The correspondence* $\tilde{R}_i : [0, \sum_{j\neq i} \max_{s_j \in S_j} h_j(s_j)] \to 2^{S_i}$ *will be single-valued except possibly at isolated points.*

Proof. $\tilde{R}_i$ has convex values since so does R_i. To arrive at a contradiction, assume that there exists an open interval $(a,b) \subseteq [0, \sum_{j\neq i} \max_{s_j\in S_j} h_j(s_j)]$ upon which $\tilde{R}_i$ is not single-valued. From upper hemi-continuity of $\tilde{R}_i$ follows that there exist $Q^a < Q^b$ both in (a,b) and an interval $[c,d] \subseteq [0, \max_{s_i\in S_i} h_i(s_i)]$ such that $[c,d] \subseteq \tilde{R}_i(Q)$ for all $Q^a \leq Q \leq Q^b$. Evidently, $x_i \in \tilde{R}_i(Q + x_i - x_i)$ hence $x_i \in B_i(Q + x_i)$, for all $x_i \in [c,d]$ and all $Q^a \leq Q \leq Q^b$. But now fix $Q \in (Q^a, Q^b)$ and $x_i \in (c,d)$ and let $M = \{x_i' \in [c,d] : Q' + x_i' = Q + x_i \text{ for some } Q' \in [Q^a, Q^b]\}$ which is a closed interval (not a point!). By construction $M \subseteq B_i(Q + x_i)$, hence for all $m \in M$: $\Psi_i(s_i, Q + x_i) = 0$ for some $s_i \in S_i$ with $h_i(s_i) = m$. But the local solvability condition implies in particular that 0 is a regular value for $\Psi_i(\cdot, Q + x_i)$ which in turn implies that the number of solutions to $\Psi_i(\cdot, Q + x_i) = 0$ is finite (Milnor (1965), p.8). But since for each m in a (convex) interval we have some $s_i \in S_i$ such that $h_i(s_i) = m$ and $\Psi_i(s_i, Q + x_i) = 0$, the equation $\Psi_i(\cdot, Q + x_i) = 0$ must have uncountably many solutions. A contradiction. ■

The next Lemma's proof is based on the implicit function theorem together with repeated use of Lemma 4.

Lemma 5 *Any selection from* $B_i(Q)$ *is locally isolated.*

Proof. $\eta_i' \in B_i(Q') \Rightarrow [\Psi_i(s_i', Q') = 0$ for some $s_i' \in S_i$ with $h_i(s_i') = \eta_i']$. By local solvability $|D_{s_i}\Psi_i(s_i', Q')| \neq 0$ which by the implicit function theorem implies the existence of a locally unique, differentiable function $f_i : (Q' - \epsilon, Q' + \epsilon) \to S_i$ such that $\Psi_i(f_i(Q), Q) = 0$ for all $Q \in (Q' - \epsilon, Q' + \epsilon)$, and such that $f_i(Q') = s_i'$. The statement of the lemma does not follow directly from this, however, because we may have two different solutions to $\Psi_i(\cdot, Q') = 0$: $\Psi_i(s_i, Q') = 0$ and $\Psi_i(\tilde{s}_i, Q') = 0$, $s_i \neq \tilde{s}_i$ with $\eta_i' \equiv h_i(s_i) = h_i(\tilde{s}_i)$. Intuitively, the problem here is that local uniqueness in terms of s_i does not (seem!) to imply local uniqueness in terms of $\eta_i = h_i(s_i)$ as stated in the lemma.

[47]Consider such $\eta_i' \in B_i(Q')$. Clearly either $\eta_i' = 0$ or $\eta_i' = \max S_i$. Take $\eta_i' = 0$ (the proof is the same in either case). Let $[Q^a, Q^b]$, $Q \in [Q^a, Q^b]$ be the maximal interval (necessarily closed) for which $\{0\} \in B_i(Q)$ all $Q \in [Q^a, Q^b]$. It is easy to see that we must have $\Psi_i(0, Q^a) = \Psi_i(0, Q^b) = 0$. This is because by varying Q either below Q^a [or above Q^b] we get a continuous, non-constant extension $b_i(Q) \in B_i(Q)$ with $b_i(Q^a) = 0$ [$b_i(Q^b) = 0$]. In particular, such an extension must lie in the interior of S_i for $Q \neq Q^a$ [$Q \neq Q^b$]. But then $\Psi_i(b_i(Q), Q) = 0$ for all $Q \neq Q^a$, and by continuity of b_i and Ψ_i follows that $\Psi_i(0, Q^a) = 0$ [$\Psi_i(0, Q^b) = 0$ respectively]. Importantly, by uniform local solvability $D_{s_i}\Psi_i(0, Q^a) < 0$ and $D_{s_i}\Psi_i(0, Q^b) < 0$. We may therefore replace Ψ_i with a function such that (i) $\Psi_i(0, Q) = 0$ for all $Q \in [Q^a, Q^b]$ and (ii) $D_{s_i}\Psi_i(0, Q) < 0$ all Q and is continuous. Note that any such "replacement" of Ψ_i is unproblematic for our arguments, except that it must be well-defined and continuously differentiable (C^1) or else our results will break down, in particular any application of the implicit function theorem requires that such a replacement is C^1 [and this is where uniform local solvability comes in: without it we might have $D_{s_i}\Psi_i(0, Q^a) > 0$ and $D_{s_i}\Psi_i(0, Q^b) < 0$ and so any replacement as above would have to "cross 0" violating the local solvability condition].

Since $s_i, \tilde{s}_i \in R_i(Q' - \eta_i')$, the above situation can of course only arise if R_i is not single-valued at $Q' - \eta_i'$. In fact, it can only happen if $\tilde{R}_i$ is not single-valued at $Q' - \eta_i'$ since otherwise $\eta_i = h_i(s_i)$ for all $s_i \in R_i(Q' - \eta_i')$ (a convex set), which definitely contradicts local uniqueness of solutions to $\Psi_i(\cdot, Q') = 0$. Now, when such multiplicity in terms of s_i arises, the implicit function theorem will give us two functions f_i and $\tilde{f}_i$ such that $\Psi_i(f_i(Q), Q) = 0$, $\Psi_i(\tilde{f}_i(Q), Q) = 0$, and $f_i(Q) \neq \tilde{f}_i(Q)$ for Q close to Q' (in addition, $f_i(Q') = s_i$ and $\tilde{f}_i(Q') = \tilde{s}_i$). Since h_i, f_i and $\tilde{f}_i$ are differentiable at Q', $Q - h_i(f_i(Q))$ and $Q - h_i(\tilde{f}_i(Q))$ will obviously be differentiable at Q'. Neither term can be constant in Q: If this were the case for, say, $Q - h_i(f_i(Q))$ we would have $h_i(f_i(Q)), \eta_i' \in \tilde{R}_i(Q - h_i(f_i(Q)) = \tilde{R}_i(Q' - \eta_i')$ where necessarily $h_i(f_i(Q)) \neq h_i(f_i(Q')) = \eta_i'$ (for $Q - h_i(f_i(Q))$ to be locally constant at Q', $f_i(Q)$ obviously cannot be locally constant at Q'). This violates the fact that $\tilde{R}_i$ can only be multi-valued at isolated points. From this follows that $Q - h_i(f_i(Q))$ $(Q - h_i(\tilde{f}_i(Q)))$ will either be strictly increasing or strictly decreasing locally at Q'. But then we can for any Z close to $Q' - \eta_i'$ find Q and $\tilde{Q}$ such that $Z = Q - f_i(Q) = \tilde{Q} - \tilde{f}_i(\tilde{Q})$, and since $f_i(Q), \tilde{f}_i(\tilde{Q}) \in \tilde{R}_i(Z)$, $\tilde{R}_i$ must be multi-valued not just at $Q' - \eta_i'$ but also at any Z close to $Q' - \eta_i'$. This once again contradicts the fact that $\tilde{R}_i$ must be singe-valued except possibly at isolated point. ■

Lemma 6 *$B_i(Q)$ consists of at most a single element.*

Proof. As is clear graphically, if $B_i(Q)$ is not single-valued for some Q, there must lie at least one point (x_i, y_i) on the graph of $\tilde{R}_i(Q - \cdot)$ $((x_i, y_i)$, $y_i \in \tilde{R}_i(Q - x_i))$ with the property that a line with slope $+1$ intersects the graph precisely at (x_i, y_i) and everywhere else in a neighborhood either lies below or above the graph. Since $y_i \in \tilde{R}_i(Q + y_i - (x_i + y_i))$, it follows that $x_i + y_i \in B_i(Q + y_i)$. But either raising or lowering Q will now lead to two continuous selections from B_i, b_i and $\tilde{b}_i$ say, both of which satisfy $b_i(Q + y_i) = \tilde{b}_i(Q + y_i)$. This contradicts lemma 5. ■

In the following, let b_i be the function such that $B_i(Q) = \{b_i(Q)\}$ (of course, $b_i(Q)$ is only well-defined if $B_i(Q) \neq \emptyset$). Let $\theta_i \equiv \max \tilde{R}_i(0)$ and $\rho_i \equiv \min \tilde{R}_i(\overline{x}_i)$ where $\overline{x}_i \equiv \sum_{j \neq i} \max_{s_j \in S_j} h_j(s_j)$. Since $\theta_i \in \tilde{R}_i(\theta_i - \theta_i)$ and $\rho_i \in \tilde{R}_i(\overline{x}_i + \rho_i - \rho_i)$, we must have $b_i(\theta_i) = \theta_i$ and $b_i(\overline{x}_i + \rho_i) = \rho_i$. It can never be the case that $\overline{x}_i + \rho_i = \theta_i$.[48] Hence the previous constructions marks two different points on the backward reply function b_i. Assume first that $\theta_i < \overline{x}_i + \rho_i$. Then the graph of $\tilde{R}_i(\overline{x}_i + \rho_i - \eta)$ must lie strictly below the 45° line for all $\eta > \rho_i$ since if not it would lie everywhere above the diagonal, which would imply that $B_i(\theta_i) = \emptyset$ (observe that we are here using that B_i is single-valued since this implies that $\tilde{R}_i(Q - \cdot)$ cannot intersection with the 45°-line twice). Likewise, the graph of $\tilde{R}_i(\theta_i - \eta)$ must lie completely above the 45°-line for $\eta < \theta_i$, otherwise we would have $B_i(\overline{x}_i - \rho_i) = \emptyset$. In case $\theta_i > \overline{x}_i + \rho_i$, the "dual" conclusions apply for the same reasons (by "dual" we mean that $\tilde{R}_i(\overline{x}_i + \rho_i - \eta)$ lies above the 45° line for $\eta > \rho_i$ and $\tilde{R}_i(\theta_i - \eta)$ lies below the 45°-line for $\eta < \theta_i$). From now on we are going to focus on the first of the above cases where $\theta_i < \overline{x}_i + \rho_i$ (all arguments immediately carry over to the case where $\theta_i > \overline{x}_i + \rho_i$).

The next three conclusions follow immediately from the fact that a change in Q, from Q to $Q + \Delta$ say, corresponds to an exact parallel shift of the graph of $\tilde{R}_i(Q - \cdot)$ either to the left $\Delta < 0$ or to the right $\Delta > 0$. First, we see that $B_i(Q) = \emptyset$ for all $Q \notin [\theta_i, \overline{x}_i + \rho_i]$. Secondly, we see that $B_i(Q) \neq \emptyset$ for all $Q \in [\theta_i, \overline{x}_i + \rho_i]$, so on this interval the function b_i is actually well-defined. Finally, we see that $\min \tilde{R}_i(Q - \eta_i) > \eta_i$ for $\eta_i < b_i(Q)$ and $\max \tilde{R}_i(Q - \eta_i) < \eta_i$ for $\eta_i > b_i(Q)$. Graphically, this last observation means that $b_i(Q)$ corresponds to a point where $\tilde{R}_i(Q - \cdot)$ intersects with the 45°-line "from above".

[48]If $\theta_i = \overline{x}_i + \rho_i$ then clearly $\rho_i < \theta_i$. In addition, $\rho_i \in \tilde{R}_i(\overline{x}_i + \rho_i - \rho_i) = \tilde{R}_i(\theta_i - \rho_i)$ hence $\theta_i, \rho_i \in B_i(\theta_i)$ contradicting that B_i is single-valued.

Let $\theta = \max_i \theta_i$ and $\delta = \min_i[\overline{x}_i + \rho_i]$. It is clear that,

$$z(Q) = \sum_i b_i(Q),$$

is a well-defined and continuous function precisely on the interval $[\theta, \delta]$ and that $z(\theta) \geq \theta$ and $z(\delta) \leq \delta$. We have suppressed t from the previous exposition. When t is included, all of the conclusions still hold of course only now we must write $z(Q,t) = \sum_i z_i(Q,t)$ and this will be well-defined for all $Q \in [\theta(t), \delta(t)]$, where both $\theta(t)$ and $\delta(t)$ are increasing in t (that these are increasing in t follow directly from the definition of these together with the definition of a positive shock). Taking $t \in T = [a,b]$, it is convenient to extend $z(\cdot, t)$ such that this is defined on $[\theta(a), \delta(b)]$ for all t. We do so simply by taking $z(Q,t) = z(\theta(t), t)$ for all $\theta(a) \leq Q < \theta(t)$ and $z(Q,t) = z(\delta(t), t)$ for $I - 1 + \rho(b) \geq Q \geq \delta(t)$. Crucially, this will not introduce any new equilibrium aggregates since $z(Q,t) = z(\theta(t), t) > Q$ for all $Q < \theta(t)$, and $Q < z(Q,t) = z(\delta(t), t)$ for all $Q > \delta(t)$. We now have:

Lemma 7 *$z(Q,t)$ is increasing in t.*

Proof. Due to the way the extension of z was made above (in particular, the fact that $\theta(t)$ and $\delta(t)$ are both increasing in t), the conclusion immediately follows if we can show that each $b_i(Q,t)$ is increasing in t. $b_i(Q,t)$ corresponds to the intersection between $\tilde{R}_i(Q - \cdot, t)$ and the 45°-line where $\tilde{R}_i(Q - \eta_i)$ is strictly above (below) the 45°-line for $\eta_i < b_i(Q)$ ($\eta_i > b_i(Q)$). By assumption, t is a positive shock in the sense that the smallest and largest selections of $\tilde{R}_i(Q - \eta_i, t)$ are increasing in t (for all fixed Q and η_i). Moreover, the smallest (respectively, the largest) selection from an upper hemi-continuous correspondence with range $\mathbb{R}$ is lower semi-continuous (respectively, upper semi-continuous).[49] In particular, the least selection is "lower semi-continuous from above" and the greatest selection is "upper semi-continuous from below". Combining we see that the correspondence $\tilde{R}_i(Q - \eta_i, t) - \{\eta_i\}$ satisfies all of the conditions of Corollary 2 in Milgrom and Roberts (1994). This allows us to conclude that $b_i(Q,t)$ is increasing in t. ■

To summarize, $Q^*(t)$ is an equilibrium aggregate given t if and only if $z(Q^*(t), t) = Q^*(t)$. In addition, we have proved that $z(Q,t)$ is continuous in Q and increasing in t. Finally, recalling the definitions of $\theta(a)$ and $\theta(b)$ from above, we have that $z : [\theta(a), \delta(b)] \times T \to \mathbb{R}$ satisfies $z(\theta(a), t) \geq \theta(a)$ and $z(\delta(b), t) \leq \delta(b)$ for all t. The conclusion of the Theorem now follows from the same argument we used at the end of the proof of the previous lemma (alternatively, it also follows from the simpler version of this result that applies to the continuous function $z(Q,t) - Q$, see *e.g.* Villas-Boas (1997)).

10.6 Proof of Theorem 7

The statement is proved only for the smallest equilibrium aggregate (the proof for the largest aggregate is similar). Let $b_{m'}$ be the least backward reply map as described in the proof of Theorem 6. Let us define $t(I+1) = s^*_{I+1} \in \mathbb{R}^N_+$ (*i.e.*, as equal to the entrant's equilibrium strategy) and $t(I) = 0$. Then $Q_*(I)$ and $Q_*(I+1)$ are the least solutions to, $Q(I) = g(b(Q(I)), t(I))$ and $Q(I+1) = g(b(Q(I+1)), t(I+1))$, respectively. Since g is increasing in t and $t(I) \leq t(I+1)$, this is a positive shock to the aggregate backward reply map. That $Q_*(I) \leq Q_*(I+1)$ must hold then follows by the same arguments as in the proof of Theorem 6. Clearly, $Q_*(I) = Q_*(I+1)$ cannot hold unless $t(I) = t(I+1)$ since g is strictly increasing. ■

[49]Let $F : X \to 2^{\mathbb{R}}$ be such a correspondence, and f_* and f^* the smallest and largest selections, *i.e.*, $f^*(x) \equiv \max_{z \in F(x)} z$ and $f_*(x) \equiv -[\max_{z \in -F(x)} z]$. Since the value function of a maximization problem is upper semi-continuous when the objective function is continuous and the constraint correspondence is upper hemi-continuous, it follows that f^* is upper semi-continuous and moreover $\tilde{f}_*(x) = \max_{z \in -F(x)} z$ is upper semi-continuous, thus implying that $f_* = -\tilde{f}_*$ is lower semi-continuous.

10.7 Proof of Proposition 1

We begin by verifying the uniform local solvability condition. Direct calculations yield

$$\Psi_i(s_i,Q) = V_i \cdot \left[\frac{h_i'(s_i)}{R+Q} - \frac{H'\left(H^{-1}(Q)\right)h_i'(s_i)h_i(s_i)}{(R+Q)^2}\right] - c_i'(s_i)\,,$$

and

$$\begin{aligned} D_{s_i}\Psi_i &= \frac{V_i h_i''(s_i)}{h_i'(s_i)} \cdot \left[\frac{h_i'(s_i)}{R+Q} - \frac{H'\left(H^{-1}(Q)\right)h_i'(s_i)h_i(s_i)}{(R+Q)^2}\right] \\ &\quad -c_i''(s_i) - V_i \frac{H'\left(H^{-1}(Q)\right)(h_i'(s_i))^2}{(R+Q)^2}. \end{aligned}$$

Therefore, when $\Psi_i(s_i,Q)=0$, we have

$$D_{s_i}\Psi_i = \frac{h_i''(s_i)}{h_i'(s_i)}c_i'(s_i) - c_i''(s_i) - V_i \cdot \frac{H'\left(H^{-1}(Q)\right)(h_i'(s_i))^2}{(R+Q)^2}.$$

Dividing both sides by $c_i'(s_i) > 0$, we conclude that $D_{s_i}\Psi_i < 0$ whenever $\Psi_i = 0$. Thus the uniform local solvability condition is satisfied.

Next, consider the payoff function of player i after the change of coordinates $s_i \mapsto z_i = h_i(s_i)$ (for $i \in \mathcal{I}$): $\tilde{\pi}_i(z) = V_i z_i \left[R + H(\sum_{j=1}^{I} z_j)\right]^{-1} - \tilde{c}_i(z_i)$. It is straightforward to verify that $\tilde{c}_i = c_i \circ h_i^{-1}$ is a convex function under our hypotheses, and using this it may be verified that $D_{z_i}\tilde{\pi}_i(z) = 0$ implies that $D^2_{z_i^2}\tilde{\pi}_i(z) < 0$. Hence any interior extremum is a strict maximum, from which follows that there is at most one extremum, necessarily a global maximum (it is possible that this maximum is on the boundary, but in this case it continues to be unique; see also Remark 3). Since the previous change of coordinates is a diffeomorphism, the previous conclusion carries over to the original payoff functions: In particular, the first-order conditions will be sufficient for a maximum which is what we need in order to apply our results (again, see Remark 3).

Now, parts 1 and 2 the proposition follow directly from Theorems 6 and 7. Part 3 follows from Theorem 8 by noting that the condition for $s_i^*(t)$ to be locally increasing in a positive shock t is

$$-[D_{s_i}\Psi_i(s_i^*, g(s^*), t)]^{-1} D_Q\Psi_i(s_i^*, g(s^*), t) \geq 0. \tag{27}$$

Since, as shown above, $D_{s_i}\Psi_i(s_i^*, g(s^*), t) < 0$, (27) holds if and only if $D_Q\Psi_i(s_i^*, Q^*, t) \geq 0$ where $Q^* = g(s^*)$. For the same reason, the condition for $s_i^*(t)$ to be decreasing in t when t does not directly affect player i (the second statement of part 3), is satisfied is and only if $D_Q\Psi_i(s_i^*, Q^*) \leq 0$.

Since,

$$\begin{aligned} &D_Q\Psi_i(s_i^*, Q^*, t) = V_i \times \\ &\left[-\frac{h_i'(s_i^*)}{(R+Q^*)^2} + \frac{2H'\left(H^{-1}(Q^*)\right)h_i'(s_i^*)h_i(s_i^*)}{(R+Q^*)^3} - \frac{H''\left(H^{-1}(Q^*)\right)(H^{-1})'(Q^*)h_i'(s_i^*)h_i(s_i^*)}{(R+Q^*)^2}\right], \end{aligned}$$

(27) will hold if and only if

$$h_i(s_i^*) \geq \eta(Q^*),$$

where

$$\eta(Q^*) \equiv \left[\frac{2H'\left(H^{-1}(Q^*)\right)}{(R+Q^*)} - \frac{H''\left(H^{-1}(Q^*)\right)}{H'\left(H^{-1}(Q^*)\right)}\right]^{-1}.$$

This shows that player i will increase its effort if it is "dominant" as defined in the proposition. If instead $h_i(s_i^*) \leq \eta(Q^*)$, *i.e.*, if the player is *not* "dominant", $D_Q\Psi_i(s_i^*, Q^*) \leq 0$ and by Theorem 8 follows that if the player is not affected by the shock, she will decrease her effort in equilibrium. ■

10.8 "Perverse" Comparative Statics Results in the Model of Section 5.4

Here we provide a specific example with strategic substitutes where the local solvability condition is violated. As will be seen, this leads to "perverse" comparative statics outcomes.

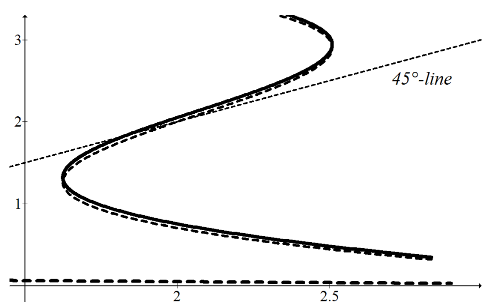

Figure 9: Backward reply correspondences (dashed). Aggregate backward reply correspondence (solid).

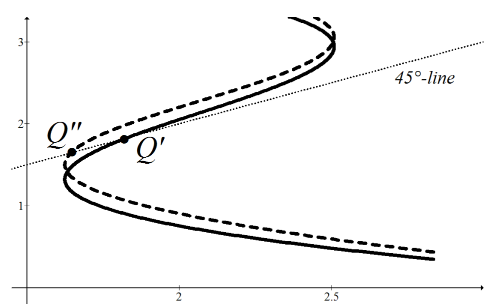

Figure 10: A decrease in marginal costs for the second firm leads to a decrease in aggregate output from Q' to Q''.

Suppose that there are only two firms and $P(Q) = K - Q$ for some constant $K > 0$. Suppose also that firm 1's costs are given by $0.5q_1^2(\alpha_1 - a_1) + C_1(a_1)$ for some differentiable, strictly increasing and strictly convex function C_1. This implies that its payoff function is

$$\pi_i(q, a) = [K - Q]q_1 - 0.5q_1^2(\alpha_1 - a_1) - C_1(a_1).$$

The first-order conditions for firm 1 can be written as,

$$K - Q - q_1 - q_1(\alpha_1 - a_1) = 0, \text{ and } 0.5q_1^2 = C_1'(a_1).$$

Since C_1 is strictly increasing, $(C_1')^{-1}$ is well-defined. $(C_1')^{-1}$ is strictly increasing since C_1 is strictly convex (and conversely, when $(C_1')^{-1}$ is strictly increasing, C_1 must be strictly convex). Let us define $G_1(z) \equiv (C_1')^{-1}(0.5z^2)$ which will also be strictly increasing. Choosing G_1 is equivalent to choosing C_1. Let

$$G_1(q_1) = -\delta_1 q_1^2 + \gamma_1 q_1 + \beta_1,$$

where $\gamma_1, \delta_1 > 0$, and $\beta_1 < 2$, so that the best responds choice of quantity for firm 1 becomes the solution to the following cubic equation:

$$K - Q + (\beta_1 - 1 - \alpha_1)q_1 - \delta_1 q_1^3 + \gamma_1 q_1^2 = 0. \tag{28}$$

Figure 9 plots q_1 as a function of Q for a particular choice of parameters (the dashes "inverse-S" shaped curve). The second dashed curve (the negatively sloped line) shows the same relationship for firm 2 (q_2 as a function of Q). Concretely, firm 2's cost function is assumed to take the form: $[\alpha_2 + 0.5\beta_2 q_2^2 - \gamma_2(a_2)^{1/2}] + \delta_2 a_2$. This yields a simple linear relationship between Q and q_2:

$0 = K - Q - \beta_2 q_2$. The solid line in the figure is the aggregate backward reply correspondence which shows $q_1 + q_2$ as a function of Q (the sum of the two dashed curves).[50]

A Cournot equilibrium is given by the solid curve's intersection with the 45°-line in Figure 9. Figure 10 depicts the same aggregate backward reply correspondence as in Figure 9 (solid), together with a similarly constructed aggregate backward reply correspondence (dashed). The only difference between the two's parameter values is that for the dashed curve β_2 is lower.[51] Naturally, a reduction in β_2 corresponds to a reduction in the marginal cost of firm 2. The figure shows that such a decrease in marginal costs reduces aggregate output Q. It can also be verified for the parameters here, the two firms' payoff/profit functions are strictly concave (even though the cost function of firm 1 is not convex).

10.9 Proof of Theorem 10

We begin with a technical lemma:

Lemma 8 *Suppose Assumption 1 holds. Then:*

(i) $\left[D_{s_i}\Psi_i\left(s_i, \sum_{j=1}^I s_j, t\right)\right]^{-1}$ *exists and all of its elements are non-positive; and*

(ii) $D^2_{s_i s_i}\Pi_i\left(s_i, \sum_{j=1}^I s_j, t\right)$ *is negative definite.*

Proof. For a matrix $\mathbf{A}$ with non-negative off-diagonal entries the following four statements are equivalent (see Berman and Plemmons (1994), pages 135-136): (1) all eigenvalues of $\mathbf{A}$ have negative real parts; (2) all real eigenvalues of $\mathbf{A}$ are negative; (3) there exists a vector $x \in \mathbb{R}^N_{++}$ such that $\mathbf{A}x \in \mathbb{R}^N_{--}$; (4) $\mathbf{A}^{-1}$ exists and all of its elements are non-positive.

It is clear from (20) that if $D^2_{s_i s_i}\Pi_i$ has non-negative off-diagonal entries and $D_Q\Psi_i$ is non-positive, then $D_{s_i}\Psi_i$ must have non-negative off-diagonal entries. By assumption, all real eigenvalues of $D_{s_i}\Psi_i\left(s_i, \sum_{j=1}^I s_j, t\right)$ are negative, hence (4) holds verifying the first claim of the lemma. For the second claim, we use that (3) holds for $D_{s_i}\Psi_i$, and let $x \in \mathbb{R}^N_{++}$ be such that $D_{s_i}\Psi_i \cdot x \in \mathbb{R}^N_{--}$. Clearly $D_Q\Psi_i \cdot x \in \mathbb{R}^N_{-}$ because $D_Q\Psi_i$ is non-positive. Hence from (20) follows that $D^2_{s_i s_i}\Pi_i\left(s_i, \sum_{j=1}^I s_j, t\right) x \in \mathbb{R}^N_{--}$. But then (since $D^2_{s_i s_i}\Pi_i$ has non-negative off-diagonal elements) all of its eigenvalues have negative real parts, and being symmetric it is therefore negative definite. ■

Next note that since $D_Q b = -[D_{s_i}\Psi_i]^{-1} D_Q\Psi_i$, part (i) of Lemma 8 implies that the backward reply function b is decreasing in Q (since $D_Q\Psi_i$ is a non-positive matrix in view of the fact that the payoff function exhibits decreasing differences).

Finally, to establish the main result, differentiate $Q = b(Q + t)$ to obtain:

$$dQ = D_Q b(Q + t)dQ + D_Q b(Q + t)dt.$$

Since $D_Q b(Q + t)$ is non-singular, this is equivalent to

$$[[D_Q b(Q + t)]^{-1} - \mathbf{I}]dQ = dt.$$

[50] The specific set of parameter values yielding the configuration in Figure 9 are: $K = 4$, $\beta_1 - 1 - \alpha_1 = -4.4$, $\gamma_1 = 2.5$, $\delta_1 = 0.4$, and $\beta_2 = 40$. Note that given these parameter values G_1 will be strictly increasing (C_1 will be strictly convex) whenever $q_1 < 3.125$. It is also straightforward to verify that any perturbation of these parameters leads to the same comparative static results, so that this perverse comparative static is "robust".

[51] Concretely, $\beta_2 = 10$ for the dashed curve and $\beta_2 = 40$ for the solid curve.

The *sufficiency* part of the theorem will follow if we can show that $dt \geq 0 \Rightarrow dQ \leq 0$. By the previous equation, this is equivalent to: $[[D_Q b(Q+t)]^{-1} - \mathbf{I}]dQ \geq 0 \Rightarrow dQ \leq 0$. An alternative (but again equivalent) way of writing this is,

$$[\mathbf{I} - [D_Q b(Q+t)]^{-1}]dQ \geq 0 \ \Rightarrow \ dQ \geq 0 \ . \tag{29}$$

The statement in (29) is very well known in matrix algebra: a matrix $\mathbf{A}$ such that $Ax \geq 0 \Rightarrow x \geq 0$ is called a *monotone matrix* (Berman and Plemmons (1994)). A well known result from matrix algebra tells us that a matrix is monotone if and only if it is a non-singular M-matrix (Berman and Plemmons (1994), page 137). Since $[\mathbf{I} - [D_Q b(Q+t)]^{-1}]$ is non-singular by assumption, it is a non-singular M-matrix when it is an M-matrix (as assumed in the theorem). Hence, it will be monotone and so any small increase in t (in one or more coordinates) will lead to a decrease in each of Q's coordinates.

As for the theorem's *necessity* statement, assume that $[\mathbf{I} - [D_Q b(Q+t)]^{-1}]$ is *not* an M-matrix. By the result just used, this is the same as saying that $[\mathbf{I} - [D_Q b(Q+t)]^{-1}]$ is *not* monotone, which implies that $dQ \not\leq 0$ and $[\mathbf{I} - [D_Q b(Q+t)]^{-1}]dQ \leq 0$ for at least one vector $dQ \in \mathbb{R}^N$. We cannot have $[\mathbf{I} - [D_Q b(Q+t)]^{-1}]dQ = 0$ since $\mathbf{I} - [D_Q b(Q+t)]^{-1}$ is non-singular; hence $[\mathbf{I} - [D_Q b(Q+t)]^{-1}]dQ < 0$ for some such vector $dQ \not\leq 0$. Now we simply pick $t'' - t' = dt = -[\mathbf{I} - [D_Q b(Q+t)]^{-1}]dQ > 0$ and the associated change in the aggregate dQ will then be increasing in at least one coordinate/component, which is the statement of the theorem. ■

10.10 Proof of Proposition 6

We begin by verifying Assumption 1. We have,

$$\Psi_i = \begin{pmatrix} P'(Q)\,q_i + P(Q) - \frac{\partial c_i(q_i,a_i)}{\partial q_i} \\ -\frac{\partial c_i(q_i,a_i)}{\partial a_i} - \frac{\partial C_i(a_i,A)}{\partial a_i} - \frac{\partial C_i(a_i,A)}{\partial A} \end{pmatrix} ,$$

and therefore,

$$D_{(q_i,a_i)}\Psi_i = \begin{pmatrix} P'(Q) - \frac{\partial^2 c_i}{\partial q_i^2} & -\frac{\partial^2 c_i}{\partial q_i \partial a_i} \\ -\frac{\partial^2 c_i}{\partial q_i \partial a_i} & -\frac{\partial^2 c_i}{\partial a_i^2} - \frac{\partial^2 C_i}{\partial a_i^2} - \frac{\partial^2 C_i}{\partial a_i \partial A} \end{pmatrix}, \text{ and}$$

$$D_{(Q,A)}\Psi_i = \begin{pmatrix} P''(Q)\,q_i + P'(Q) & 0 \\ 0 & -\frac{\partial^2 C_i}{\partial a_i \partial A} - \frac{\partial^2 C_i}{\partial A^2} \end{pmatrix}.$$

The convexity of cost functions and condition 1 in the proposition ensure that the eigenvalues of $D_{(q_i,a_i)}\Psi_i$ are both real and negative, so strong solvability is satisfied.[52] Strategic substitutes holds since, in view of condition 2, $D_{(Q,A)}\Psi_i$ is a non-positive matrix, and the sum of $D_{(Q,A)}\Psi_i$ and $D_{(q_i,a_i)}\Psi_i$ has a non-negative diagonal. This verifies that the technology adoption game satisfies Assumption 1.

To apply (the sufficiency part of) Theorem 10, we must establish that the matrix $\mathbf{I} - [D_Q b(Q+t)]^{-1}$ is an M-matrix. In the case of two aggregates, this is the case when the determinant of

[52] In particular,

$$D_{(q_i,a_i)}\Psi_i = \begin{pmatrix} -\frac{\partial^2 c_i}{\partial q_i^2} & -\frac{\partial^2 c_i}{\partial q_i \partial a_i} \\ -\frac{\partial^2 c_i}{\partial q_i \partial a_i} & -\frac{\partial^2 c_i}{\partial a_i^2} \end{pmatrix} + \begin{pmatrix} P' & 0 \\ 0 & -\frac{\partial^2 C_i}{\partial a_i^2} - \frac{\partial^2 C_i}{\partial a_i \partial A} \end{pmatrix},$$

and each of these is negative semidefinite and one of them is negative definite.

$D_Q b(Q,t)$ is positive. By (22), $D_Q b(Q,t) = \sum_{j=1}^{I} D_Q b_j(Q,t)$, and since the sum of negative quasi-definite matrices is negative quasi-definite (see Remark 8), a sufficient condition for $D_Q b(Q,t)$ to be an M-matrix is that each of the matrices $D_Q b_j(Q,t)$ is negative quasi-definite. Using (21):

$$D_Q b_i(Q,t) = -[D_{s_i}\Psi_i(b_i(Q,t),Q,t)]^{-1} D_Q \Psi_i(b_i(Q,t),Q,t) =$$

$$-\frac{1}{\Delta}\begin{pmatrix} (-\frac{\partial^2 c_i}{\partial a_i^2} - \frac{\partial^2 C_i}{\partial a_i^2} - \frac{\partial^2 C_i}{\partial a_i \partial A})(P''(Q)\,q_i + P'(Q)) & \frac{\partial^2 c_i}{\partial q_i \partial a_i}(-\frac{\partial^2 C_i}{\partial a_i \partial A} - \frac{\partial^2 C_i}{\partial A^2}) \\ \frac{\partial^2 c_i}{\partial q_i \partial a_i}(P''(Q)\,q_i + P'(Q)) & (P'(Q) - \frac{\partial^2 c_i}{\partial q_i^2})(-\frac{\partial^2 C_i}{\partial a_i \partial A} - \frac{\partial^2 C_i}{\partial A^2}) \end{pmatrix}$$

Where $\Delta > 0$ denotes the determinant of $D_{(q_i,a_i)}\Psi_i$. Moreover, $D_Q b_i$ is negative quasi-definite if and only if $[D_Q b_i + (D_Q b_i)^T]/2$ is negative definite. This will be the case if and only if the determinant of $[D_Q b_i + (D_Q b_i)^T]/2$ is positive, or equivalently, if and only if

$$(\frac{\partial^2 c_i}{\partial a_i^2} + \frac{\partial^2 C_i}{\partial a_i^2} + \frac{\partial^2 C_i}{\partial a_i \partial A})(-P' + \frac{\partial^2 c_i}{\partial q_i^2})(P'' q_i + P')(-\frac{\partial^2 C_i}{\partial a_i \partial A} - \frac{\partial^2 C_i}{\partial A^2}) >$$

$$(\frac{\partial^2 c_i}{\partial q_i \partial a_i})^2[\frac{1}{2}(P'' q_i + P' - \frac{\partial^2 C_i}{\partial a_i \partial A} - \frac{\partial^2 C_i}{\partial A^2})]^2.$$

A sufficient (though not necessary) condition for this inequality to hold is that:

$$\frac{\frac{\partial^2 c_i}{\partial a_i^2}\frac{\partial^2 c_i}{\partial q_i^2}}{(\frac{\partial^2 c_i}{\partial q_i \partial a_i})^2} \geq \frac{[\frac{1}{2}(P'' q_i + P' - \frac{\partial^2 C_i}{\partial a_i \partial A} - \frac{\partial^2 C_i}{\partial A^2})]^2}{(P'' q_i + P')(-\frac{\partial^2 C_i}{\partial a_i \partial A} - \frac{\partial^2 C_i}{\partial A^2})},$$

which is ensured by condition 3 in the proposition.

The remaining conclusions of the Proposition follow directly from Theorems 11 and 12. ■

10.11 Proof of Theorem 13

We begin with Theorem 6. To simplify the notation in the proof, let us focus on the case with a linear aggregator, $g(s) = \sum_{j=1}^{I} s_j$ (the general case is proved by the exact same argument, the only difference is that it becomes very notation-heavy). Then, for each i, $G_i(Q,y) = Q - y$. Define $M_i(y,t) \equiv \arg\max_{Q \geq y} \Pi_i(Q-y,Q)$. Clearly, $M_i(y,t) - \{y\} = \tilde{R}_i(y,t)$, where $\tilde{R}_i$ is the "reduced" best response correspondence (*i.e.*, best response as a function of the sum of the opponents' strategies). Hence, we can write $M_i(Q - s_i,t) - \{Q\} = \tilde{R}_i(Q - s_i,t) - \{s_i\}$. Given single-crossing, $M_i(y,t)$ is ascending in y (*e.g.*, Milgrom and Shannon (1994), Theorem 4). Therefore, $\tilde{R}_i(Q - s_i,t) - \{s_i\}$ must be descending in s_i. Moreover, $\tilde{R}_i(Q - s_i,t) - \{s_i\}$ is convex-valued. Let $B_i(Q,t) = \{s_i \in S_i : s_i \in \tilde{R}_i(Q - s_i,t)\}$. $B_i(Q,t) \neq \emptyset$ since: $z \in \tilde{R}_i(Q - \perp_i,t) \Rightarrow z - \perp_i \geq 0$ (where $\perp_i \equiv \min S_i$), while $z \in \tilde{R}_i(Q - \top_i,t) \Rightarrow z - \top_i \leq 0$.[53] It may be verified that $B_i(Q,t)$ is always a convex set (possibly a singleton), and that the least and greatest selections of $B_i(Q,t)$ are increasing in t (this follows from the same argument used in the proof of Lemma 7).

[53]Here it is necessary to extend $\tilde{R}_i$ by defining $\tilde{R}_i(z) = \{\tilde{R}_i(0)\}$ when $z < 0$, where we have here taken $\perp_j = 0$ for all j so that the least value Q can assume is 0. It is clear that with this extension $\tilde{R}_i(Q - s_i) - \{s_i\}$ is (still) descending and now always passes through 0. Importantly, the extensions (one for each agent), do not introduce any new fixed points for $B = \sum_j B_j$: Given Q, if $s_i \in \tilde{R}_i(Q - s_i)$, then either $Q - s_i \geq 0$ or $Q < s_i \in \tilde{R}_i(Q - s_i)$. But if $s_i > Q$ for just one i, we cannot have $\sum_j s_j = Q$.

We can now again use the argument used in the proof of Lemma 7, in order to conclude that the smallest and largest fixed points of the convex valued, upper hemi-continuous, and ascending in t correspondence $B(Q,t) = \sum_j B_j(Q,t)$, are increasing in t. This establishes Theorem 6. The proof of Theorem 7 applies to the present case (the only difference being that we now have a convex-valued aggregate backward reply correspondence, $B = \sum_j B_j$). Finally, Theorem 8 follows from the implicit function theorem together with Theorem 6. ■

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