Aggregate Comparative Statics[☆]

Daron Acemoglu[a], Martin Kaae Jensen[b]

[a]*Department of Economics, Massachusetts Institute of Technology (e-mail: daron@mit.edu).*
[b]*Department of Economics, University of Birmingham (e-mail: m.k.jensen@bham.ac.uk).*

Abstract

In aggregative games, each player's payoff depends on her own actions and an aggregate of the actions of all the players. Many common games in industrial organization, political economy, public economics, and macroeconomics can be cast as aggregative games. This paper provides a general and tractable framework for comparative static results in aggregative games. We focus on two classes of games: (1) aggregative games with strategic substitutes and (2) nice aggregative games, where payoff functions are continuous and concave in own strategies. We provide simple sufficient conditions under which positive shocks to individual players increase their own actions and have monotone effects on the aggregate. The results are illustrated with applications to public good provision, contests, Cournot competition and technology choices oligopoly.

Keywords:
Aggregative games, Strategic substitutes, Nice games, Comparative statics, Contests, Patent races, Public good provision, Cournot model.

1. Introduction

In aggregative games, each player's payoff depends on her own actions and some aggregate of all players' actions. Numerous games studied in the literature can be cast as aggregative games including models of competition (Cournot and Bertrand with or without product differentiation), patent races, models of contests and fighting, public good provision games, and models with aggregate demand externalities.[1] In this paper, we provide a simple general framework for comparative static analysis in aggregative games (thus generalizing Corchón [7] which is discussed in greater detail below). Our approach is applicable to a diverse set of applications that can be cast as aggregative games and enables us to provide sufficient conditions for a rich set of comparative static results.

We present results for two sets of complementary environments. First, we focus on aggregative games with strategic substitutes. In games with strategic substitutes, each player's payoff function is supermodular in her own strategy and exhibits decreasing differences in her own strategy and the strategy vector of other players. Second, we turn to "nice" aggregative games, where payoff functions are continuous, concave (or pseudo-concave) in own strategies, and twice continuously differentiable. For such games, we prove a number of results under a condition which we refer to as *local solvability* which ensures the local invertibility of the backward reply correspondence described further below.

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[1]For a long list of examples of aggregative games see Alos-Ferrer and Ania [3]. For more specific applications, see *e.g.* Cornes and Hartley [8, 10], Kotchen [21], and Fraser [14]. Issues of evolutionary stability (Alos-Ferrer and Ania [3], Possajennikov [31]), evolution of preferences (Kockesen et al. [20]), existence and stability (Dubey et al. [13] and Jensen [18]), and uniqueness of equilibrium (Cornes and Hartley [9]) have also been studied fruitfully in the context of aggregative games.

An informal summary of our results from both aggregative games with strategic substitutes and from nice aggregative games is that, under a variety of reasonable economic conditions, comparative statics are "regular" (for example, in Cournot oligopoly, a reduction in the marginal cost increases a firm's output). More precisely, in nice aggregative games with local solvability, a "positive shock" to any subset of players — defined as a change in parameters that increase the marginal payoff of the subset of players — increases the aggregate, and entry of a new player also increases the aggregate. In aggregative games with strategic substitutes, a positive shock to a player increases that player's strategy and reduces the aggregate of the remaining players' strategies, and entry of a new player reduces the aggregate of the strategies of remaining players. In addition, in aggregative games with strategic substitutes, the aggregate varies monotonically with what we call "shocks that hit the aggregator" which are changes in parameters that have a direct (positive) impact on the aggregator. In a separate section (Section 5), we illustrate all of these results in a variety of economic models, highlighting both the broad applicability of the methods we propose and adding several new results and insights.

We should emphasize at this point that there is no guarantee in general that intuitive and unambiguous comparative static results should hold in aggregative games. Take an increase in a player's marginal payoff such as a reduction in an oligopolist's marginal cost: Even though the *first-order effect* of such a shock will of course be positive, it is possible that *higher-order effects* go in the opposite direction so that in equilibrium, the player ends up lowering her strategy and the aggregate falls (see Acemoglu and Jensen [2] for an example of this kind). In this light, a major contribution of our paper is to provide minimal conditions to ensure that such higher-order effects do not dominate so that comparative statics becomes "regular". In particular, our first set of theorems shows that such "perverse" outcomes cannot arise in aggregative games with strategic substitutes, and our second set of results establishes that they can be ruled out in nice aggregative games by the local solvability condition mentioned above.

Our paper is related to a number of different strands in the literature. Comparative static results in most games are obtained using the implicit function theorem. The main exception is for supermodular games (games with strategic complements). Topkis [40, 41], Milgrom and Roberts [26] and Vives [46] provide a framework for deriving comparative static results in such games. These methods do not extend beyond supermodular games.

More closely related to our work, and in many ways its precursor, is Corchón [7]. Corchón [7] provides comparative static results for aggregative games with strategic substitutes, but only under fairly restrictive conditions, which, among other things, ensure uniqueness of equilibria. In contrast, our comparative static results for aggregative games with strategic substitutes are valid without any additional assumptions. Another similarity between our paper and Corchón [7] is that both make use of the so-called *backward reply correspondence* of Selten [37]. In an aggregative game, the backward reply correspondence gives the (best response) strategies of players that are compatible with a given value of the aggregate.[2] In a seminal paper, Novshek [30] used this correspondence to give the first general proof of existence of pure-strategy equilibria in the Cournot model without assuming quasi-concavity of payoff functions (see also Kukushkin [22]). Novshek's result has since been strengthened and generalized to a larger class of aggregative games (*e.g.,* Dubey et al. [13] and Jensen [18]), and our results on games with strategic substitutes utilize Novshek [30]'s construction in the proofs.[3] Our results on nice aggregative games blend the backward reply approach with the equilibrium comparison results reported in Milgrom and Roberts [27] and Villas-Boas [44].

An alternative to working directly with backward reply correspondences as we do, is to use "share correspondences" introduced by Cornes and Hartley [8]. The share correspondence is the backward reply

[2]The first systematic study of aggregative games (German: *aggregierbaren Spiele*) can be found in Selten [37]. After defining aggregative games, Selten proceeds to define what he calls the *Einpassungsfunktion* (Selten [37], p. 154), that is, the backward reply function of an individual player. As Selten proves, the backward reply correspondence is single-valued (a function) provided that the player's best-response function has slope greater than -1. The assumptions imposed by Corchón [7] imply that the slope of players' best-response functions lie strictly between -1 and 0, so that the backward reply correspondence is both single-valued and decreasing. Neither is necessarily the case in many common games and neither is imposed in this paper.

[3]Novshek's explicit characterization of equilibria is similar to the characterization of equilibrium in supermodular games that uses the fixed point theorem of Tarski [39]. Both of these enable the explicit study of the behavior of "largest" and "smallest" fixed points in response to parameter changes. Tarski's result is used, for example, in the proof of Theorem 6 in Milgrom and Roberts [26].

correspondence divided with the aggregate. This transformation of the problem is useful for questions related to uniqueness and existence and can be used for explicitly characterizing the equilibrium and deriving comparative statics directly in certain cases. However, transforming backward reply correspondences in this way does not simplify any arguments in this paper or strengthen any results.[4]

The rest of the paper is organized as follows. Section 2 defines aggregative games, equilibrium, and backward reply correspondences. Section 3 provides the general comparative static results for aggregative games with strategic substitutes. Section 4 presents our results for nice games under the local solvability condition. Section 5 shows how the results from Sections 3 and 4 can be used to obtain general characterization results in various applications, including games of private provision of public goods, contests, Cournot competition and technology choice in oligopoly. Section 6 concludes and the Appendix contains some proofs omitted from the text.

2. Aggregative Games

In this paper we study non-cooperative games $\Gamma_t = ((\pi_i, S_i)_{i\in\mathcal{I}}, t)$ with finite sets of players $\mathcal{I} = \{1, \ldots, I\}$, finite-dimensional strategy sets $S_i \subseteq \mathbb{R}^N$, and payoff functions $\pi_i : S \times \{t\} \to \mathbb{R}$. $t \in T \subseteq \mathbb{R}^M$ is a vector of exogenous parameters, and the basic comparative statics question we wish to address is how the set of equilibria of Γ_t varies with t. As usual we define $S \equiv \prod_{i=1}^{I} S_i$ and $S_{-i} \equiv \prod_{j\neq i} S_j$ with typical elements $s \in S$ and $s_{-i} \in S_{-i}$. Throughout S is assumed to be compact and each payoff function $\pi_i : S \times T \to \mathbb{R}$ is assumed to be upper semi-continuous on $S \times T$ and continuous on $S_{-i} \times T$. These assumptions ensure that the best-reply correspondences $R_i(s_{-i}, t) \equiv \arg\max_{s_i\in S_i} \pi_i(s_i, s_{-i}, t)$ will be non-empty, compact valued, and upper hemi-continuous.

Recall, *e.g.,* from Gorman [15], that a function $g : S \to \mathbb{R}$ is *additively separable* if there exist strictly increasing functions $H, h_1, \ldots, h_I : \mathbb{R} \to \mathbb{R}$ such that $g(s) = H(\sum_{i\in\mathcal{I}} h_i(s_i))$ for all $s \in S$. The unweighted sum $g(s) = \sum_{i\in\mathcal{I}} s_i$ and the mean $g(s) = I^{-1}\sum_{i\in\mathcal{I}} s_i$ are obvious examples. In fact all of the standard means, including the harmonic mean, the geometric mean, and the power means are additive separable functions (Jensen [18], section 2.3.2). Two other important examples are $g(s) = (\alpha_1 s_1^{\beta} + \ldots + \alpha_I s_I^{\beta})^{1/\beta}$, $S \subseteq \mathbb{R}^N_+$, and $g(s) = \prod_{i\in\mathcal{I}} s_i^{\alpha_i}$, $S \subseteq \mathbb{R}^N_{++}$, where $\beta, \alpha_1, \ldots, \alpha_I > 0$, which are, respectively, a CES function and a Cobb-Douglas function.[5]

Definition 1. ***(Aggregative Games)*** *The game $\Gamma_t = ((\pi_i, S_i)_{i\in\mathcal{I}}, t)$ is aggregative if there exists a continuous and additively separable function $g : S \to X \subseteq \mathbb{R}$ (the aggregator) and functions $\Pi_i : S_i \times X \times \{t\} \to \mathbb{R}$ (the reduced payoff functions) such that for each player $i \in \mathcal{I}$:*

$$\pi_i(s_i, s_{-i}, t) = \Pi_i(s_i, g(s), t) \text{ for all } s \in S \tag{1}$$

Clearly, an aggregative game is fully summarized by the tuple $((\Pi_i, S_i)_{i\in\mathcal{I}}, g, t)$. The definition of an equilibrium is standard:

Definition 2. ***(Equilibrium)*** *Let $((\Pi_i, S_i)_{i\in\mathcal{I}}, g, t)$ be an aggregative game. Then $s^*(t) = (s_1^*(t), \ldots, s_I^*(t))$ is a (pure-strategy Nash) equilibrium if for each player $i \in \mathcal{I}$,*

$$s_i^*(t) \in \arg\max_{s_i\in S_i} \Pi_i(s_i, g(s_i, s_{-i}^*), t).$$

When $s^*(t)$ is an equilibrium, $Q(t) \equiv g(s^*(t))$ is called an equilibrium aggregate given t. And if smallest and largest equilibrium aggregates exist, these are denoted by $Q_*(t)$ and $Q^*(t)$, respectively.

[4] It is straightforward to recast Novshek's original existence argument in terms of share correspondences (by simply dividing through everywhere with the aggregate Q). Similarly one would be able to recast our proofs for games with strategic substitutes in terms of share correspondences, but this does not lead to any simplification. As for our results on "nice" games, these are based on the idea that under the local solvability condition, the aggregate backward reply correspondence will be a continuous single-valued function. This obviously holds for the aggregate backward reply correspondence if and only if it holds for the associated share correspondence (since the latter's values equals the former's divided with Q). But this construction does not simplify or enrich our analysis; it simply restates our results in a somewhat different language.

[5] In the first case $h_i(s_i) = \alpha_i s_i^{\beta}$ (with $s_i \geq 0$) and $H(z) = z^{1/\beta}$. In the second $h_i(s_i) = \alpha_i \log(s_i)$ and $H(z) = \exp(z)$ (with $s_i > 0$).

Aggregative games with additively separable aggregators are studied in detail in Cornes and Hartley [11] and section 2.3.2 in Jensen [18]. This class is more general than that studied by Selten [37] and Corchón [7] who consider only the case where $g(s) = \sum_i s_i$. More general classes of aggregative games have also been proposed in the literature (*e.g.* Jensen [18], Martimort and Stole [25]) and questions related to existence, best-response potentials, and stability can be addressed more generally than under Definition 1.[6] Aggregative games are also closely related to semi-anonymous games which are games where each player's payoff depends on his own strategy and the *distribution* of opponents' strategies (see, *e.g.,* Kalai [19]).[7]

Since opponents' strategies enter player i's payoff function only through the aggregator $g(s) = H(\sum_i h_i(s_i))$, player i's best-reply correspondence can always be expressed as,

$$R_i(s_{-i}, t) = \tilde{R}_i(\sum_{j \neq i} h_j(s_j), t). \tag{2}$$

In words, a player's best replies will always be a function of the *aggregate of the other players* $\sum_{j \neq i} h_j(s_j)$, and the exogenous parameter t. We refer to $\tilde{R}_i$ as the *reduced best-reply correspondence*. Now fix an *aggregate*, *i.e.*, a value in the domain of the aggregator, $Q \in X \equiv \{g(s) : s \in S\}$; and note that $Q = g(s)$ $\Leftrightarrow \sum_{j \neq i} h_j(s_j) = H^{-1}(Q) - h_i(s_i)$. Substituting into the right-hand side of (2) we can find the set of best replies, for each player $i \in \mathcal{I}$, that are consistent with Q:

$$B_i(Q, t) \equiv \{s_i \in S_i : s_i \in \tilde{R}_i(H^{-1}(Q) - h_i(s_i), t)\}. \tag{3}$$

$B_i : X \times T \to 2^{S_i} \cup \emptyset$ is the *backward reply correspondence* of player i. It is obvious that any value of the aggregator $Q(t)$ for which $Q(t) = g(s^*(t))$ and $s_i^*(t) \in B_i(Q(t), t)$ for all i, will induce an equilibrium in accordance with Definition 2. Consequently $Q(t)$ is an equilibrium aggregate given t if and only if $Q(t) \in Z(Q(t), t)$ where $Z : X \times T \to 2^X \cup \emptyset$ is the *aggregate backward reply correspondence* defined by:

$$Z(Q, t) \equiv \{g(s) \in X : s_i \in B_i(Q, t) \text{ for all } i \in \mathcal{I}\} \tag{4}$$

As mentioned, the basic question we address in this paper is how the set of equilibria of Γ_t vary with t, and more specifically our main focus is on how the equilibrium aggregates vary with t. Throughout, under the assumed conditions, equilibria, and therefore equilibrium aggregates, will not be unique, and therefore our comparative static statements will be similar in spirit to the results of Milgrom and Roberts [27] and tell us that "the smallest and largest equilibrium aggregates are increasing in t". Intuitively, this means that the *set* of all equilibrium aggregates is contained in an interval $[Q_*(t), Q^*(t)]$ whose lower and upper bounds are increasing in t.[8] If additional conditions are imposed (or hold in a specific application) which ensure that $Q_*(t) = Q^*(t)$, then our results of course describe the behavior of this unique equilibrium aggregate.

3. Aggregative Games with Strategic Substitutes

In this section we show that if an aggregative game has strategic substitutes then regular comparative statics can be obtained under the very weak compactness and continuity conditions introduced at the beginning of Section 2. So just as in games with strategic complementarities (Vives [46], Milgrom and Roberts [26], Topkis [42]), no differentiability, quasi-concavity, or convexity conditions are needed to obtain

[6] As shown by Cornes and Hartley [11], Definition 1 represents the most general class of aggregative games that admits backward reply correspondences when there are three or more players and the aggregator is strictly increasing. In the working paper version of this paper (Acemoglu and Jensen [2]) we address the situation where the function H in the definition of an aggregator is merely assumed to be increasing (rather than strictly increasing) which still allows for backward reply correspondences.

[7] Clearly, an aggregative game is semi-anonymous if the aggregator g is symmetric. If g is not symmetric, an aggregative game will not be semi-anonymous except in pathological cases (such as when payoff functions are constant in $g(s)$). Semi-anonymous games play a central role in the study of *large* games, and not surprisingly, aggregative games similarly yield very nice results when there is a continuum of players (Acemoglu and Jensen [1]).

[8] Put differently, the set of fixed points of Z will be ascending in t (see *e.g.* Topkis [42]). See also Milgrom and Roberts [27] for a general methodological discussion.

comparative statics results. Note that the aggregative structure is critical for this observation. In the general class of games with strategic substitutes much more restrictive assumptions are needed in order to obtain meaningful comparative statics results (see Roy and Sabarwal [32]). Concrete applications of the results can be found in Section 5, in particular that section contains an application to a game where strategy sets are multidimensional which illustrates the results' full scope.

The definition of a game with strategic substitutes is standard.

Definition 3. ***(Strategic Substitutes)*** *The game $\Gamma_t = ((\pi_i, S_i)_{i\in\mathcal{I}}, t)$ is a game with strategic substitutes if strategy sets are lattices and each player's payoff function $\pi_i(s_i, s_{-i}, t)$ is supermodular in s_i and exhibits decreasing differences in s_i and s_{-i}.*

Equivalently, we will also say that a game has (or features) strategic substitutes. S_i is a *lattice* if $s, s' \in S_i$ implies $s \wedge s', s \vee s' \in S_i$ where $s \wedge s'$ and $s \vee s'$ denote, respectively, the infimum and supremum of s and s'. $\pi_i : S_i \times S_{-i} \times T \to \mathbb{R}$ is *supermodular* in s_i if for all fixed $(s_{-i}, t) \in S_{-i} \times T$: $\pi_i(s_i \vee s'_i, s_{-i}, t) - \pi_i(s_i, s_{-i}, t) \geq \pi_i(s'_i, s_{-i}, t) - \pi_i(s_i \wedge s'_i, s_{-i}, t)$ for all $s_i, s'_i \in S_i$. Finally, $\pi_i : S_i \times S_{-i} \times T \to \mathbb{R}$ exhibits *decreasing differences* in s_i and s_{-i} if for all $t \in T$ and $s'_i > s_i$: $\pi_i(s'_i, s_{-i}, t) - \pi_i(s_i, s_{-i}, t)$ is non-increasing in s_{-i} (see *e.g.*, Topkis [40] or Topkis [42]). As we explain in a moment, all three are normally straight-forward to verify in aggregative games. It follows directly from Topkis' theorem (Topkis [40]) that in a game of strategic substitutes, each player's best-response correspondence will be decreasing in the strong set order in opponents' strategies. That best-response correspondences are decreasing is the essential property used in our proofs and our results remain valid under any set of conditions that ensure this outcome.[9] For a detailed exposition of the general class of games with strategic substitutes, see *e.g.* Roy and Sabarwal [33].

A game that is both aggregative and has strategic substitutes is an *aggregative game with strategic substitutes*. Definition 3 is usually straightforward to verify in aggregative games. Particularly simple is the case with a linear aggregator $g(s) = \sum_{j=1}^{I} s_j$ where $\pi_i(s_i, s_{-i}, t) = \Pi_i(s_i, \sum_{j=1}^{I} s_j, t)$. If π_i is twice differentiable, then decreasing differences is equivalent to non-positive cross-partials, *i.e.,* to having $D^2_{s_i s_j}\pi_i \leq 0$ for all $j \neq i$. But since $D^2_{s_i s_j}\pi_i = D^2_{12}\Pi_i + D^2_{22}\Pi_i$ for *all* $j \neq i$, we see that the aggregative structure implies that decreasing differences holds for player i under the *single* condition that:[10]

$$D^2_{12}\Pi_i + D^2_{22}\Pi_i \leq 0 \tag{5}$$

Since a game with one-dimensional strategy sets automatically satisfies the lattice and supermodularity conditions of Definition 3, a game with one-dimensional strategy sets and a linear aggregator will consequently be an aggregative game with strategic substitutes if and only if (5) holds for all $i \in \mathcal{I}$. The strict inequality version of (5) is part of what Corchón [7] calls "strong concavity" (Corchón [7], Assumption 2).

Theorem 1. ***(Existence)*** *Let $((\Pi_i, S_i)_{i\in\mathcal{I}}, g, t)$ be an aggregative game with strategic substitutes. Then there exists an equilibrium $s^*(t) \in S$, and also smallest and largest equilibrium aggregates $Q_*(t)$ and $Q^*(t)$. Moreover, $Q_* : T \to \mathbb{R}$ is a lower semi-continuous function and $Q^* : T \to \mathbb{R}$ is an upper semi-continuous function.*

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[9] As a consequence, instead of Definition 3 (supermodularity and decreasing differences), we could equivalently work with *quasi-supermodularity* and the *dual single-crossing property* of Milgrom and Shannon [28]. Note also that since players do not take Q as given, there is no exact relationship between strategic substitutes and the condition that $\Pi_i(s_i, Q)$ exhibits decreasing differences in s_i and Q. For example, suppose that $N = 1$, $g(s) = \sum_{j=1}^{I} s_j$, and assume that payoff functions are twice differentiable. Then the requirement for strategic substitutes is $D^2_{s_i s_q}\Pi_i(s_i, \sum_{j=1}^{I} s_j) = D^2_{12}\Pi_i(s_i, Q) + D^2_{22}\Pi_i(s_i, Q) \leq 0$ where $Q = \sum_{j=1}^{I} s_j$. Decreasing differences in s_i and Q, on the other hand, requires that $D^2_{12}\Pi_i(s_i, Q) \leq 0$. Clearly neither condition implies the other.

[10] The derivatives in these statements are defined by $D^2_{12}\pi_i(s_i, \sum_{j\neq i} s_j, t) = \frac{\partial^2 \Pi_i(x,y,t)}{\partial x \partial y}_{|(x,y)=(s_i, \sum_{j\neq i} s_j)}$ and $D^2_{22}\pi_i(s_i, \sum_{j\neq i} s_j, t) = \frac{\partial^2 \Pi_i(x,y,t)}{\partial^2 y}_{|(x,y)=(s_i, \sum_{j\neq i} s_j)}$.

Proof. Definition 1 is a special case of the class of quasi-aggregative games of Jensen [18] (see Jensen [18], section 2.3.2. for an explicit verification of this claim). As a consequence, aggregative games as defined in this paper are either best-reply potential games or best-reply pseudo-potential games when best-reply correspondences have decreasing selections, and thus they have a pure strategy Nash equilibrium (Jensen [18], Corollary 1).

To prove the claims concerning the smallest and largest equilibrium aggregates, first define $\text{Gr}[R_i] : T \to 2^S$ such that $\text{Gr}[R_i](t) \equiv \{s \in S : s_i \in R_i(s_{-i}, t)\}$ for each player i. This correspondence is upper hemi-continuous and has a closed graph since if $s_i^m \in R_i(s_{-i}^m, t^m)$ for a convergent sequence $(s^m, t^m) \to (s, t)$, then by the fact that R_i itself has a closed graph, $s_i \in R_i(s_{-i}, t)$. Let $E(t) = \cap_i \text{Gr}[R_i](t)$ denote the set of equilibria given $t \in T$. Since $E : T \to 2^S$ is defined as the intersection of a finite number of upper hemi-continuous correspondences, it is itself upper hemi-continuous. Since $E(t) \subseteq S$, where S is compact, E therefore also has compact values. The existence of the smallest and largest equilibrium aggregates, $Q_*(t) = \min_{s \in E(t)} g(s)$ and $Q^*(t) = \max_{s \in E(t)} g(s)$, therefore follows from Weierstrass' theorem since g is continuous. Upper semi-continuity of $Q^* : T \to \mathbb{R}$ follows directly from the fact that g is upper semi-continuous and E is upper hemi-continuous (see Ausubel and Deneckere [5], Theorem 1). Lower semi-continuity of Q_* follows by the same argument since $Q_*(t) \equiv -\max_{s \in E(t)} -g(s)$ and g is also lower semi-continuous. ■

Naturally, pure-strategy equilibria are not necessarily unique under the conditions of Theorem 1, so in general $Q_*(t)$ is different from $Q^*(t)$. For conditions that guarantee uniqueness in games with strategic substitutes see *e.g.* Theorem 2.8. in Vives [47] or Corchón [7]. See also the discussion of uniqueness in Section 4.1.1.

Our first substantive result addresses the situation where an exogenous parameter $t \in T \subseteq \mathbb{R}$ directly "hits" the aggregator in the following sense:

Definition 4. ***(Shocks that Hit the Aggregator)*** *Consider the payoff function $\pi_i = \pi_i(s_i, s_{-i}, t)$. Then an increase in $t \in T \subseteq \mathbb{R}$ is said to be a shock that hits the aggregator if* (1) *can be strengthened to:*

$$\pi_i(s, t) = \Pi_i(s_i, G(g(s), t)) \text{ for all } s \in S, \tag{6}$$

where $G = G(g(s), t)$ is continuous, increasing in t and additively separable in s and t.

Note that the terminology adopted here requires some care and elaboration. First, there is a direction of change in the definition: a shock that hits the aggregator actually "hits it positively" as implied by the condition that G is increasing in t, but we drop the "positively" qualifier to simplify the terminology. Of course, if G were decreasing in t, $-t$ would be a shock that hits the aggregator. Second, a shock that hits the aggregator does not change the aggregator g or the aggregate $g(s)$, it merely changes the aggregator directly in the payoff function. For example, if $\pi_i(s, t) = \Pi_i(s_i, t + \sum_{j=1}^I s_j)$, then an increase in t is a shock that hits the aggregator as seen by taking $G(g(s), t) = t + g(s)$ and $g(s) = \sum_{j=1}^I s_j$. And clearly, changing t does not change g or $g(s)$. Examples of shocks that hit the aggregator include an increase in the state's provision of the public good in the public good provision model (Section 5.1), an increase in the discount factor in a contest/patent race (Section 5.2), or a downwards shift in the demand curve in the Cournot model (Section 5.3).

Notice that when a shock hits the aggregator, the marginal payoff of each player *decreases* (provided that marginal payoffs are defined).[11] Hence we would intuitively expect a shock that hits the aggregator to lead to a *decrease* in the aggregate. The next theorem shows that in an aggregative game with strategic substitutes, this is indeed the case.

[11] By strategic substitutes, agent i's marginal payoff must be decreasing in opponents' strategies and hence, since G is increasing in s and t, an increase in t must lead to a decrease in marginal payoff.

Theorem 2. ***(The Comparative Statics of Shocks that Hit the Aggregator)*** *In an aggregative game with strategic substitutes, a shock that hits the aggregator leads to a decrease in the smallest and largest equilibrium aggregates, i.e., the functions $Q_*(t)$ and $Q^*(t)$ will be decreasing in $t \in T$.*

Proof. Let $t \in T$ be a shock that hits the aggregator. In particular then $\pi_i(s,t) \equiv \Pi_i(s_i, G(g(s), t))$ for all i, where $G(g(s),t) = \tilde{H}(h_T(t) + \sum_{j=1}^{I} h_j(s_j))$ and $g(s) = H(\sum_{i\in\mathcal{I}} h_i(s_i))$ is the aggregator. The reduced best-reply correspondence (2) can therefore be written as: $R_i(s_{-i},t) \equiv \tilde{R}_i(h_T(t)+\sum_{j\neq i} h_j(s_j))$. In what follows we abuse notation slightly and define $Q = \sum_{i\in\mathcal{I}} h_i(s_i)$ and speak of this as the aggregate. Since the true aggregate $H(\sum_{i\in\mathcal{I}} h_i(s_i))$ increases if and only if $\sum_{i\in\mathcal{I}} h_i(s_i)$ increases, the conclusion that Q decreases with t obviously implies that conclusion of the theorem. Given Q define the following correspondence for each $i \in \mathcal{I}$:

$$\tilde{B}_i(Q,t) \equiv \{\eta \in h_i(S_i) : \eta \in h_i \circ \tilde{R}_i(h_T(t) + Q - \eta)\} \tag{7}$$

If strategy sets are one-dimensional and the aggregator is linear, this is just the usual backward reply correspondence B_i as defined in Section 2 (in particular, $\sum_{i\in\mathcal{I}} h_i(s_i)$ is the true aggregate in this case). The corresponding aggregate backward reply correspondence is then:

$$Z(Q,t) \equiv \sum_{i=1}^{I} \tilde{B}_i(Q,t) \tag{8}$$

As explained in Section 2, Q will be an equilibrium aggregate given t if and only if Q is a fixed point for this correspondence ($Q \in Z(Q,t)$).

Let $q(Q,t) \in Z(Q,t)$ be the "Novshek-selection" shown as the thick segments in the figure below. Further details about this selection can be found in the appendix (Section 7.1). As is clear from the figure, the Novshek selection has two key features in games with strategic substitutes: Firstly, it will be decreasing in Q, *i.e.*, if $Q' \geq Q$ then $q(Q',t) \leq q(Q,t)$. Secondly, its left end-point $Q^{\min}$ will be an equilibrium aggregate, *i.e.*, $q(Q^{\min},t) = Q^{\min}$. The latter of these claims is proved in Novshek [30] (see also Kukushkin [22]) in the case of a linear aggregator. Since their proofs carry over directly to our slightly more general setting, we omit the details. It is clear that without strategic substitutes, one would generally not be able to find a selection with these properties — in particular $Q^{\min}$ might not be an equilibrium aggregate. Note also that as is clear in the figure, $Q^{\min}$ must necessarily be the *largest equilibrium aggregate*. The reason is that if $Q^* \in Z(Q^*,t)$ and $Q^* > Q^{\min}$, Condition *1.* in the definition of the Novshek selection (see Definition 10 in Secton 7.1 for further details) would be violated.

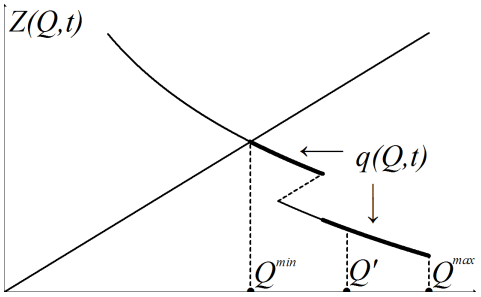

Figure 1: Constructing the aggregate "Novshek selection"

We are now ready to prove the main claim of the theorem, namely that the largest equilibrium aggregate $Q^{\min}$ characterized above, will be decreasing in t (for the case of the smallest equilibrium aggregate, see the end of the proof). To make the proof more accessible, we first illustrate it graphically, followed by formal

arguments in all cases. It is sufficient to establish this result for all local changes in t since if a function is decreasing at all points, it is globally decreasing (of course, the associated equilibrium aggregate may well jump — the argument is local only in regard to changes in t).

First, note that since h_T is an increasing and continuous function, any selection $\hat{q}(Q,t)$ from Z will be locally decreasing in Q if and only if it is locally decreasing in t (this is an immediate consequence of the separability assumptions — see the definition of $\tilde{B}_i$ above). Likewise, such a selection $\hat{q}(Q,t)$ will be locally continuous in Q if and only if it is locally continuous in t. Figures 2-5 illustrate the situation for $t' < t''$. The fact that the direction of the effect of a change in Q and t is the same accounts for the arrows drawn. In particular, any increasing segment on the graph of Z will be shifted up when t is increased, and any decreasing segment will be shifted down.

There are four cases: Either the graph of Z's restriction to a neighborhood of $Q^{\min}$ is locally continuous and locally decreasing in Q and t (Case I) or locally continuous and locally increasing in Q and t (Case II). Otherwise, continuity does not obtain which is the same as saying that the equilibrium aggregate must "jump" when t is either increased from t' to t'' (Cases III and IV) or decreased from t'' to t' [If t is decreased, case III reduces to Case I and Case IV reduces to Case II.]

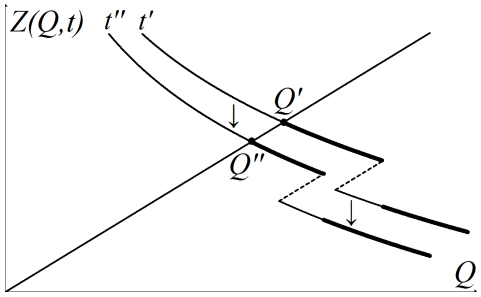

Figure 2: Case I

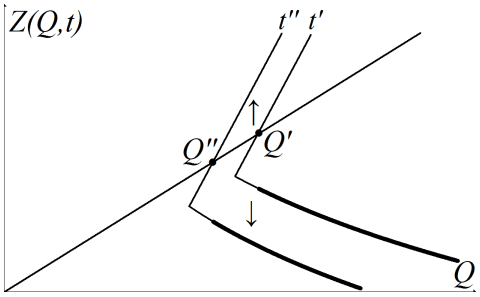

Figure 3: Case II

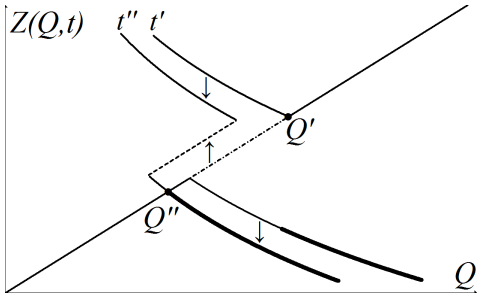

Figure 4: Case III

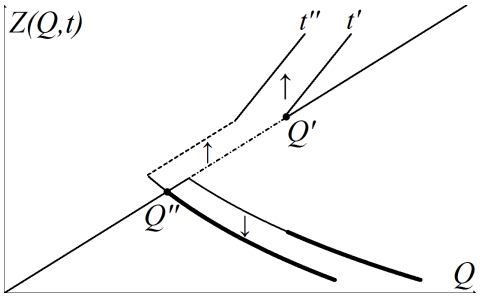

Figure 5: Case IV

Cases III and IV are easily dealt with: If the equilibrium aggregate jumps, it necessarily jumps *down* (and so is decreasing in t). The reason is that an increase in t will always correspond to the graph of Z being shifted to "the left" (*i.e.,* any increasing segment will be shifted up, and any decreasing segment shifted down which was the formulation used above). Hence no new equilibrium above the original largest one can appear, the jump has to be to a lower equilibrium as is also immediate in light of the figures. We now consider the more difficult Cases I and II in turn. Throughout the function $\hat{q}$ denotes the restriction of Z to a neighborhood of $Q^{\min}$, and Q' and Q'' refer to the equilibrium aggregate $Q^{\min}$ associated with t' and t'', respectively.

Case I: In this case there exist $\underline{Q} < \overline{Q}$ such that $\hat{q}(\underline{Q}, t) - \underline{Q} > 0$ and $\hat{q}(\overline{Q}, t) - \overline{Q} < 0$, and such that the new equilibrium aggregate Q'' lies in the interval $[\underline{Q}, \overline{Q}]$. Since $\hat{q}$ is decreasing in t, it immediately follows that $Q'' \leq Q'$ as desired. Note that this observation does not depend on continuity of $\hat{q}$ in Q, but merely on the fact that a new equilibrium aggregate Q'' exists and lies in a neighborhood of Q' in which $\hat{q}$ is decreasing (in other words, given that $\hat{q}$ is decreasing, it depends solely on the fact that the aggregate does *not* "jump").

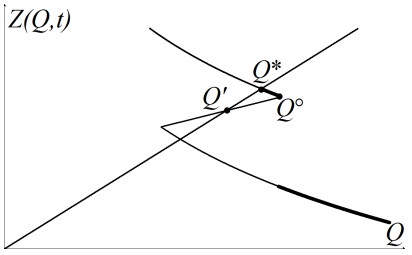

Figure 6: Slope below 1 is impossible: Q' being largest equilibrium aggregate violates that $q(Q,t)$ is decreasing in Q.

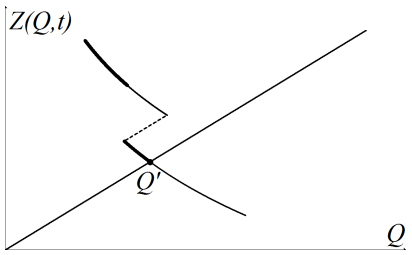

Figure 7: The "Novshek selection" leading to the smallest equilibrium aggregate.

Case II: When $\hat{q}$ is (locally) increasing, there *must* exist $\underline{Q} < Q' < \overline{Q}$ such that $\underline{Q} - \hat{q}(\underline{Q}, t) > 0$ and $\overline{Q} - \hat{q}(\overline{Q}, t) < 0$. Intuitively, this means that the slope of $\hat{q}$ is greater than 1 at the point Q' as illustrated in Figure 3. Formally, this can be proved as follows: Assume that there exists $Q^\circ > Q'$ such that $Q^\circ - \hat{q}(Q^\circ, t) \leq 0$ (intuitively this means that the slope is below unity, see Figure 6). Then since $\hat{q}(Q^\circ, t) \geq Q^\circ > Q'$, no Novshek selection could have reached Q' and there would consequently have to be a larger equilibrium Q^*, which is a contradiction.

We now prove that the equilibrium aggregate is decreasing in t: $Q'' \leq Q'$. As in the previous case, we prove this without explicit use of continuity (the proof is straightforward if continuity is used directly as seen in Figure 3). In particular, let us establish the stronger statement that $C(t) \equiv h_T(t) + Q(t)$ is decreasing in t where $Q(t)$ is the largest equilibrium aggregate given t (since $h_T(t)$ is increasing in t, it is obvious that $Q(t)$ must be decreasing in t if $C(t)$ is decreasing). Define the following function: $f(C,t) = C - h_T(t) - \hat{q}(C - h_T(t), t)$. Clearly $C(t) = h_T(t) + Q(t)$ as defined with $Q(t)$ an equilibrium if and only if $f(C(t), t) = 0$. Let $\underline{C} = h_T(t) + \underline{Q}$ and $\overline{C} = h_T(t) + \overline{Q}$. From the previous paragraph, $f(\underline{C}, t) = \underline{Q} - \hat{q}(\underline{Q}, t) > 0$ and $f(\overline{C}, t) = \overline{Q} - \hat{q}(\overline{Q}, t) < 0$. Since $B_i(C - h_T(t), t)$ is independent of t (t cancels out in the definition of the backward reply correspondence), $\hat{q}(C - h_T(t), t)$ must be constant in t, *i.e.*, $\hat{q}(C - h_T(t), t) = \tilde{q}(C)$ for some function $\tilde{q}$ which is increasing (since we are in Case II). So f can be written as $f(C,t) = C - h_T(t) - \tilde{q}(C)$ where $\tilde{q}$ is increasing, and consequently f will be decreasing in t and Q. Considering the solution to $f(C,t) = 0$ given t, *i.e.*, $C(t)$, it immediately follows that if t increases then $C(t)$ must decrease. This finishes the proof of the claim in Case II.

Combining the previous observations, we conclude that the largest equilibrium aggregate is decreasing in t as claimed in the theorem. None of the previous conclusions depend on continuity of q in Q, and it is straightforward to verify that the same conclusions hold regardless of whether Q lies in a convex interval or not (strategy sets could be discrete, see Kukushkin [22] for the details of how the backward reply selection is constructed in such non-convex cases). The statement for the smallest equilibrium aggregate can be shown by an analogous argument. In particular, instead of considering the selection $q(Q,t)$ one begins with Q sufficiently low and studies the backward reply correspondence above the 45° line, now choosing for every Q the smallest best response (Figure 7). This completes the proof of Theorem 2. ■

Notice how the proof of Theorem 2 exploits the constructive nature of Novshek [30]'s existence proof (suitably generalized to fit the present framework). This explicit description of the largest (and smallest) equilibrium aggregates is ultimately what allows us to determine the direction of any change resulting from

a shock that increases the aggregator.

Theorem 2 is also *robust* and *global* in the sense discussed by Milgrom and Roberts [27] and Milgrom and Shannon [28]. In particular, it imposes only minimal qualitative restrictions, and allows us to deal with situations where the equilibrium aggregates "jump" when t is changed.

In many games — a classical example being that of Cournot oligopoly (*e.g.* Seade [36]) — it is interesting to study what happens to equilibria as additional players enter the game (see Alos-Ferrer and Ania [3] for a general discussion of these issues in an aggregative games framework). To formalize this, consider initially a game with I players as studied above, and now imagine that an additional player (the *entrant*) is added. The entrant player $I+1$ is (by definition) assigned the "inaction" strategy $\inf S_{I+1}$ before entry (*e.g.,* when $S_{I+1} = [0, \bar{s}]$, inaction corresponds to "zero", $\inf S_{I+1} = 0$; zero production in oligopoly, say, or zero contribution to the provision of a public good). If we take as aggregator $g(s) = g(s_1, \ldots, s_I, s_{I+1})$, this leads to a well-defined aggregative game both *before* and *after* entry: before entry there are I strategic players and s_{I+1} is just a constant, after entry this is an $I+1$ player aggregative game in the usual sense.[12]

Theorem 3. ***(The Comparative Statics of Entry)*** *In an aggregative game with strategic substitutes, entry of an additional player leads to a decrease in the smallest and largest aggregates of the existing players in equilibrium.*

Proof. This result follows from Theorem 2 by observing that the entry of an additional player corresponds to a shock that hits the aggregator. To see this, let $g^{I+1}(s_1, \ldots, s_I, s_{I+1})$ be the aggregator in the game after entry. Since g^{I+1} is additively separable, we necessarily have $g^{I+1}(s_1, \ldots, s_I, s_{I+1}) = G(g^I(s_1, \ldots, s_I), s_{I+1})$ (Vind and Grodal [45]) where G and g^I satisfy the above requirements for an increase in s_{I+1} to be a shock that hits the aggregator. Since $g^I(s_1, \ldots, s_I)$ is the aggregate of the existing players, the theorem's conclusion follows from Theorem 2. ■

Intuitively, Theorem 3 shows that in an aggregative games with strategic substitutes, entry "crowds out" existing players. While intuition may suggest that entry should make the overall aggregate inclusive of the entrant increase, it is well known from the Cournot model that this is not a general feature of games with strategic substitutes (see Seade [36], Corchón [7]). We return to this topic in the next section.

The next theorem presents our most powerful result for games with strategic substitutes. This result can be viewed as aggregative games with strategic substitutes' counterpart to the well-known monotonicity results for games with strategic complementarities (Vives [46], Milgrom and Roberts [26]). One difference, however, is that with strategic substitutes, the results apply only to shocks that are *idiosyncratic*, *i.e.*, to changes in a parameter t_i that affects a *single* player, $i \in \mathcal{I}$. Note that when $T_i \subseteq \mathbb{R}^M$ with $M > 1$, an increase in t_i means that one or more of the coordinates of t_i are increased.

Definition 5. ***(Positive Idiosyncratic Shocks)*** *An increase in $t_i \in T_i$ is a positive idiosyncratic shock to player i if $\pi_i = \pi_i(s_i, s_{-i}, t_i)$ exhibits increasing differences in s_i and t_i and if $\pi_j(s, t_i) = \pi_j(s)$ for all $j \neq i$.*

The previous definition parallels standard definitions in games with strategic complementarities (*e.g.*, Vives [47]). When π_i is twice differentiable, it will exhibit increasing differences in s_i and t_i if and only if the matrix of cross-partials is nonnegative, *i.e.,* $D^2_{s_i t_i}\pi_i \in \mathbb{R}^{N \times M}_+$ for all s and t_i. It should be mentioned that the single-crossing property of Milgrom and Shannon [28] can replace increasing differences in the previous definition without changing the following result. In the following statement, the smallest and largest equilibrium strategies for player i are defined analogously to the smallest and largest equilibrium aggregates.

[12] When the aggregator is a so-called *generalized symmetric aggregator* (Alos-Ferrer and Ania [3], see also Jensen [18], section 2.3.1) it is possible to define a game with an arbitrary number of players very elegantly. Unfortunately, that definition does not generalize to aggregators that are merely assumed to be additively separable which is why we have chosen to formulate entry as a situation with $I+1$ players one of whom is initially a "dummy" player. Needless to say, our setting extends to the entry of any number of players by repeated application of Theorem 3.

Theorem 4. ***(The Comparative Statics of Idiosyncratic Shocks)*** *In an aggregative game with strategic substitutes, a positive idiosyncratic shock to player $i \in \mathcal{I}$ leads to an increase in the smallest and largest equilibrium strategies for player i, and to a decrease in the associated aggregates of the remaining players (which are, respectively, the largest and smallest such aggregates).*

Proof. Let $\tilde{R}_i$ denote the reduced backward reply correspondence defined in Section 2. Assume without loss of generality that the idiosyncratic shock affects the first player, in particular then $\tilde{R}_i$ is independent of the shock t_1 for all $i \neq 1$. Any equilibrium will satisfy: $s_1 \in \tilde{R}_1(\sum_{j\neq 1} h_j(s_j), t_1)$ and $h_i(s_i) \in h_i \circ \tilde{R}_i(\sum_{j\neq i} h(s_j))$ for $i = 2, \ldots, I$. Consider the last $I-1$ inclusions, and rewrite these as:

$$h_i(s_i) \in h_i \circ \tilde{R}_i((\sum_{j\neq i,1} h_j(s_j)) + h_1(s_1)) \text{ for } i = 2, \ldots, I. \tag{9}$$

For any $s_1 \in S_1$, Theorem 1 implies that there exist smallest and largest aggregates of players 2 to I, $y_*(s_1)$ and $y^*(s_1)$ such that $y_*(s_1) = \sum_{j\neq 1} h_j(s_{j,*})$ and $y^*(s_1) = \sum_{j\neq 1} h_j(s_j^*)$, where the strategies are solutions to the $I-1$ inclusions in (9). And by Theorem 2, y_*, $y^* : S_1 \to \mathbb{R}$ will be decreasing functions. Now replace s_1 with $\bar{s}_1 = -s_1$, and consider:

$$\bar{s}_1 \in -\tilde{R}_1(y, t_1)$$

and

$$y \in \{y_*(-\bar{s}_1), y^*(-\bar{s}_1)\}.$$

This system is ascending in $(\bar{s}_1, y)$ and descending in t_1 in the sense of Topkis [42], hence its smallest and largest fixed points are decreasing in t_1. Therefore, the smallest and largest equilibrium strategies for player 1 are increasing in t_1, while the associated aggregates of the remaining players are decreasing in t_1. That the smallest and largest strategies for player 1 do in fact correspond to the smallest and largest strategies in the original game is easily verified: Clearly, $y_*(s_1)$ and $y^*(s_1)$ are the smallest and largest aggregates of the remaining players across all strategy profiles compatible with an equilibrium given s_1. And since $\tilde{R}_1$ is descending in y, the corresponding equilibrium strategies of player 1 must therefore necessarily be the largest and the smallest such strategies as well. ■

Section 5 contains multiple applications of this result. It immediately follows, for example, that — assuming *only* strategic substitutes — a decrease in the marginal cost of a firm in Cournot oligopoly will make that firm increase its output at the expense of the other firms. Since the Cournot model has strategic substitutes if it is merely assumed that inverse demand is concave and decreasing (Vives [47]), this conclusion is valid for *arbitrary* cost functions. This shows that Theorem 4 is a substantial generalization of existing results such as those of Corchón [7].

We end this section with a simple corollary to Theorem 4 characterizing the effects of a positive shock on payoffs:

Corollary 1. ***(Payoff Effects)*** *Assume in addition to the conditions of Theorem 4 that all payoff functions are decreasing [respectively, increasing] in opponents' strategies and that player i's payoff function is increasing [respectively, decreasing] in the idiosyncratic shock t_i. Then an increase in t_i increases [respectively, decreases] player i's payoff in equilibrium and decreases [respectively, increases] the payoff of at least one other player.*

Proof. Consider $t_i' < t_i''$ in T_i and let s' be the equilibrium given t_i' corresponding to player i's smallest strategy and s'' the equilibrium given t_i'' corresponding to player i's smallest strategy (the proof to follow is the same for the largest equilibrium strategies of player i). Under the assumptions of the Corollary, $\pi_i(s_i', g(s'), t_i') \leq \pi_i(s_i', g(s_i', s_{-i}'', t_i'')) \leq \pi_i(s_i'', g(s''), t_i'')$. Since the strategy of at least one player $j \neq i$

must decrease, the aggregate of opponents' strategies faced by that player $\sum_{k\neq j} h_k(s'_k)$ must increase (the best-response correspondences are decreasing in the strong set order). Consequently, $\pi_j(s''_j, g(s'')) \leq \pi_j(s''_j, g(s''_j, s'_{-j})) \leq \pi_j(s'_j, g(s'))$. ■

4. Nice Aggregative Games

In the previous section we saw how a number of robust and global comparative statics results can be established for aggregative games with strategic substitutes. From Vives [46], Milgrom and Roberts [26], and many others, we know that robust and global results can similarly be established in games with strategic complementarities (and these results obviously still apply if the game is aggregative). But of course, not all aggregative games feature strategic substitutes or complementarities, examples in this "neither-or" category being contests (Section 5.2) as well as public good provision models when the private good is inferior for some levels of income (Section 5.1). In this section, we present a third alternative, named the *local solvability condition* under which robust comparative statics statements can be derived in aggregative games. Unlike results on strategic substitutes or complements, the local solvability condition only works in what we call *nice games* which are in essence games that satisfy a standard battery of differentiability, convexity, and boundary conditions.[13] In nice games, the local solvability condition places sufficient structure on the aggregate backward reply correspondence of Section 2 for the equilibrium comparison results of Milgrom and Roberts [27] and Villas-Boas [44] to apply. So just as in the previous section, we obtain global and robust results which could not have been obtained by purely local methods such as a standard application of the implicit function theorem.

We begin by defining nice games. Recall that a differentiable function π_i is pseudo-concave (Mangasarian [24]) in s_i if for all $s_i, s'_i \in S_i$:

$$(s'_i - s_i)^T D_{s_i}\pi_i(s_i, s_{-i}, t) \leq 0 \;\Rightarrow\; \pi_i(s'_i, s_{-i}, t) \leq \pi_i(s_i, s_{-i}, t).$$

Naturally, any concave function is pseudo-concave.

Definition 6. ***(Nice Aggregative Games)*** *An aggregative game $((\Pi_i, S_i)_{i\in\mathcal{I}}, g, t)$ is said to be a nice aggregative game if:*

1. *The aggregator g is twice continuously differentiable.*
2. *Each strategy set S_i is compact and convex, and every payoff function $\pi_i(s,t) = \Pi_i(s_i, g(s), t)$ is twice continuously differentiable, and pseudo-concave in the player's own strategies.*
3. *For each player, the first-order conditions hold whenever a boundary strategy is a (local) best response, i.e., $D_{s_i}\Pi_i(s_i, g(s), t) = 0$ whenever $s_i \in \partial S_i$ and $(v - s_i)^T D_{s_i}\Pi_i(s_i, g(s), t) \leq 0$ for all $v \in S_i$.*

Note that part *3.* of this definition does not rule out best responses on the boundary of a player's strategy set. Instead, it simply requires first-order conditions to be satisfied whenever a best response is at the boundary. Consequently, it is weaker than the standard "Inada-type" conditions ensuring that best responses always lie in the interior of strategy sets. We show below that the boundary condition *3.* can be dispensed with if the local solvability condition is strengthened (Definition 8).

The next theorem establishes existence of an equilibrium and of smallest and largest equilibrium aggregates in any nice aggregative game.

Theorem 5. ***(Existence)*** *Let $((\Pi_i, S_i)_{i\in\mathcal{I}}, g, t)$ be a nice aggregative game. Then there exists an equilibrium $s^*(t) \in S$, and also smallest and largest equilibrium aggregates $Q_*(t)$ and $Q^*(t)$. Moreover, $Q_* : T \to \mathbb{R}$ is a lower semi-continuous function and $Q^* : T \to \mathbb{R}$ is an upper semi-continuous function.*

[13] As we show below, boundary conditions are *not* needed when strategy sets are one-dimensional.

Proof. Existence follows straight from Kakutani's fixed point theorem since best-reply correspondences will be upper hemi-continuous and have convex values (any pseudo-concave function is quasi-concave). The claims concerning the smallest and largest equilibrium aggregates follow from the same argument as in the proof of Theorem 1. ■

Just as in the setting with strategic substitutes, uniqueness of equilibrium is not implied by any of this section's conditions (including the local solvability condition). If, however, uniqueness can be established so that $Q_*(t) = Q^*(t)$ for all t, the statements to follow will just refer to the unique equilibrium aggregate.

We are now ready to introduce the *local solvability condition*, which is the key assumption of this section. Using that g is differentiable and additively separable so that $g(s) = H(\sum_j h_j(s_j))$, the marginal payoff for player i can be expressed as,

$$D_{s_i}\pi_i(s,t) = D_1\Pi_i(s_i, g(s), t) + D_2\Pi_i(s_i, g(s), t)H'(H^{-1}(g(s))Dh_i(s_i), \tag{10}$$

where $D_m\Pi_i(s_i, g(s), t) \equiv D_{x_m}\Pi_i(x_1, x_2, t)_{|(x_1,x_2)=(s_i,g(s))}$, $m = 1, 2$.

Equation (10) shows that in an aggregative game, the marginal payoff is always a function of the player's own strategy s_i, the aggregate $g(s)$, and the exogenous parameters t. Define a function $\Psi_i : S_i \times X \times T \to \mathbb{R}^N$ that makes this relationship explicit:[14]

$$\Psi_i(s_i, Q, t) \equiv D_1\Pi_i(s_i, Q, t) + D_2\Pi_i(s_i, Q, t)H'(H^{-1}(Q))Dh_i(s_i). \tag{11}$$

In the special case where strategy sets are one-dimensional and $g(s) = \sum_i s_i$, Ψ_i is precisely the function introduced by Corchón [7] in his analysis of aggregative games.[15] Naturally $\Psi_i(s_i, Q) = 0 \Leftrightarrow [D_{s_i}\pi_i(s,t) = 0$ and $g(s) = Q]$, hence when the game is nice so that first-order conditions are necessary and sufficient for an optimum:

$$s_i \in B_i(Q,t) \Leftrightarrow \Psi_i(s_i, Q, t) = 0\ , \tag{12}$$

where $B_i : X \times T \to 2^{S_i}$ is the backward reply correspondence of Section 2. It is this simple relationship between Ψ_i and B_i that makes the class of nice games valuable. If we fix Q and t and differentiate Ψ_i with respect to s_i, we get an $N \times N$ matrix $D_{s_i}\Psi_i(s_i, Q, t) \in \mathbb{R}^{N\times N}$. The determinant of this matrix is denoted by $|D_{s_i}\Psi_i(s_i, Q, t)| \in \mathbb{R}$. If strategy sets are one-dimensional, $|D_{s_i}\Psi_i(s_i, Q, t)| = D_{s_i}\Psi_i(s_i, Q, t) \in \mathbb{R}$. We are now ready to define the local solvability condition.

Definition 7. ***(Local Solvability)*** *Player $i \in \mathcal{I}$ is said to satisfy the local solvability condition if $\Psi_i(s_i, Q, t) = 0 \Rightarrow |D_{s_i}\Psi_i(s_i, Q, t)| \neq 0$ for all $s_i \in S_i$, $Q \in X$, and $t \in T$.*

As mentioned above, the following stronger version of local solvability will allow us to dispense with any boundary conditions if strategy sets are one-dimensional.

Definition 8. ***(Uniform Local Solvability)*** *When $S_i \subseteq \mathbb{R}$, player $i \in \mathcal{I}$ is said to satisfy the uniform local solvability condition if $\Psi_i(s_i, Q, t) = 0 \Rightarrow D_{s_i}\Psi_i(s_i, Q, t) < 0$ for all $s_i \in S_i$, $Q \in X$, and $t \in T$.*

Before discussing the interpretation of these conditions, it is useful to consider a concrete example. Take the Cournot model where $\Pi_i(s) = s_iP(\sum_j s_j) - c_i(s_i)$ and so $\Psi_i(s_i, Q) = P(Q) + s_iP'(Q) - c_i'(s_i)$ (suppressing here exogenous parameters). Hence the local solvability condition will hold if either $D_{s_i}\Psi_i(s_i, Q) = P'(Q) - c_i''(s_i) < 0$ or $D_{s_i}\Psi_i(s_i, Q) = P'(Q) - c_i''(s_i) > 0$ whenever $P(Q) + s_iP'(Q) - c_i'(s_i) = 0$. If the first of the two holds whenever $P(Q) + s_iP'(Q) - c_i'(s_i) = 0$, the uniform local solvability condition is satisfied. For example,

[14] Here the variables s_i and Q are *independent* arguments in Ψ_i, so that Q is kept *fixed* when taking the derivative of Ψ_i. Hence, *e.g.*, $D_{s_i}\Psi_i(s_i, Q, t) = \frac{\partial\Psi_i(x_1,x_2,t)}{\partial x_1}_{|(x_1,x_2)=(s_i,Q)}$ when $N = 1$.

[15] Corchon denotes this function by T_i (Corchón [7], page 155). See also Section 2 in Cornes and Hartley [11] for the general case considered here.

this will be the case when costs are convex and inverse demand is strictly decreasing (these conditions are clearly not necessary).[16] Other examples where the uniform local solvability condition is satisfied is in the public good provision model if the public good is strictly normal (Section 5.1), and in contests if players' cost functions have larger curvature than their success functions (Section 5.2).

The local solvability condition requires that the determinant of $D_{s_i}\Psi_i$ is nonzero on the subspace where $\Psi_i = 0$. This implies that we can *solve* the equation $\Psi_i(s_i, Q) = 0$ locally for s_i given Q, hence its name. To be precise, when $\Psi_i(s_i, Q, t) = 0$, the local solvability condition allows us to apply the implicit function theorem to conclude that there exist open neighborhoods $\mathcal{N}_{s_i} \subseteq S_i$ and $\mathcal{M}_Q \subseteq X$ of s_i and Q, respectively, and a continuous map $b_i : \mathcal{M}_Q \to \mathcal{N}_{s_i}$ such that for each $\hat{Q} \in \mathcal{M}_Q$, $b_i(\hat{Q})$ is the unique solution to $\Psi_i(s_i, \hat{Q}, t) = 0$ in $\mathcal{N}_{s_i}$. In terms of backward replies, we see from (12) that this implies local uniqueness as well as continuity of backward replies (in fact the function $b_i : \mathcal{M}_Q \to \mathcal{N}_{s_i}$ will be such a local selection from the backward reply correspondence). In each of the proofs below, this is the critical component and our results would go through under any set of conditions that ensures this outcome.

If strategy sets are one-dimensional ($S_i \subseteq \mathbb{R}$ for all i), and more generally if strategy sets are lattices and payoff functions supermodular in own strategies, we can define *positive shocks* in the standard way known from games with strategic complementarities (see *e.g.* Vives [47]). Recall again that if $t \in T \subseteq \mathbb{R}^M$, $M > 1$, then an increase in t means that at least one of t's coordinates increases.[17]

Definition 9. ***(Positive Shocks)*** *Consider the payoff functions $\pi_i = \pi_i(s_i, s_{-i}, t)$. Then an increase in t is called a positive shock to $t \in T$ if each S_i is a lattice, and π_i is supermodular in s_i and exhibits increasing differences in s_i and t. In particular, if $S_i \subseteq \mathbb{R}$ for all i, then t is a positive shock if each π_i exhibits increasing differences in s_i and t.*

Notice that if an increase in t is an idiosyncratic shock ($\pi_i = \pi_i(s, t)$ and $\pi_j = \pi_j(s)$ for all $j \neq i$), Definition 9 trivially holds for all $j \neq i$ which brings us back to Definition 5 of the previous section. The discussion immediately prior to Theorem 2 also implies that a shock that hits the aggregator is a negative shock to t ($-t$ is a positive shock) if the game features strategic substitutes, and by the same reasoning, t will be a positive shock if the game features strategic complementarities. These observations clarify how the following theorem complements our findings from the previous section.

Theorem 6. ***(Aggregate Comparative Statics)*** *Consider a nice aggregative game where each player's payoff function satisfies the local solvability condition. Then a positive shock to $t \in T$ leads to an increase in the smallest and largest equilibrium aggregates, i.e., the functions $Q_*(t)$ and $Q^*(t)$ will be increasing in t. When strategy sets are one-dimensional and each player's payoff function satisfies the uniform local solvability condition, the result remains valid without imposing the boundary condition 3. of Definition 6.*

Proof. See Section 7.2 ■

Theorem 6 is this section's main result. Since the proof is rather long, it is relegated to the Appendix. The main idea is to apply Theorem 1 of Milgrom and Roberts [27] or Theorem 1 of Villas-Boas [44] to the aggregate backward reply correspondence which, crucially, can be shown to be *continuous* and *single-valued* when the local solvability condition holds.[18] As discussed at the beginning of this section, Theorem 6 is

[16] It is worth noting that the condition $P'(Q) - c_i''(s_i) < 0$ is one of Hahn [16]'s two conditions for local stability of Cournot equilibrium (see Vives [46] Chapter 4 for an extensive discussion of this and related conditions). The other condition is strategic substitutes which is *not* needed for the following results. As mentioned by Corchón [7] (p. 156), Corchon's "strong concavity condition" reduces precisely to the two Hahn conditions in the Cournot model (except that strategic substitutes is strengthened to strict strategic substitutes). As a consequence, in Section 5.3 we generalize Corchón [7]'s results for the Cournot model. See also the discussion immediately prior to Theorem 1.

[17] This definition uses the lattice and supermodularity conditions for clarity. Positive shocks can be defined more generally without necessitating any modification in our results, but this would be at the expense of additional notation. See the working paper version of this paper (Acemoglu and Jensen [2]).

[18] Without local solvability, the aggregate backward reply correspondence may easily fail to be single-valued. And more generally it may easily fail to have selections that are "continuous but for jumps up" in the sense of Milgrom and Roberts [27]. Note that all of these statements remain valid even if best-reply correspondences are continuous and single-valued, in particular, single-valued best-replies certainly does not imply single-valued backward replies.

particularly useful in applications that feature neither strategic substitutes or strategic complementarities such as contests or patent races. In Section 5.2 we present an application of Theorem 6 to this class of models. Finally note that Theorem 6 — just as our results on strategic substitutes — is global and robust in the sense of Milgrom and Roberts [27]. In particular, the theorem applies even if an equilibrium selection "jumps"when t is raised.

Our next result is the analogue of Theorem 3 for nice aggregative games.

Theorem 7. ***(The Comparative Statics of Entry)*** *Under the conditions of Theorem 6 entry of an additional player increases the smallest and largest equilibrium aggregates, i.e., if $Q_*(I)$ and $Q^*(I)$ denote the smallest and largest equilibrium aggregates in a game with $I \in \mathbb{N}$ players then $Q_*(I) \leq Q_*(I+1)$ and $Q^*(I) \leq Q^*(I+1)$ for all $I \in \mathbb{N}$. The previous inequalities will be strict if the entrant does not choose the "inaction" strategy* $\inf S_{I+1}$.

Proof. The statement is proved only for the largest equilibrium aggregate (the proof for the smallest aggregate is similar). Consider the game with $I+1$ players but where player $I+1$ is a "dummy player" with a fixed strategy $s_{I+1} \in S_{I+1}$. Since s_{I+1} is exogenous we can define the largest aggregate backward reply map $\overline{q}(Q, s_{I+1})$ as in Section 7.2 (so in terms of that section's notation, we have here taken $t := s_{I+1}$). Note that $\overline{q}(Q, s_{I+1})$ must be strictly coordinatewise increasing in s_{I+1}. Let s^*_{I+1} denote the entrant's strategy after entry in the equilibrium associated with the largest equilibrium aggregate $Q^*(I+1)$. Then $Q^*(I)$ and $Q^*(I+1)$ are the largest solutions to, $Q(I) = f^{-1}(\overline{q}(f(Q(I)), \inf S_{I+1}))$ and $Q(I+1) = f^{-1}(\overline{q}(f(Q(I+1)), s^*_{I+1}))$, respectively (here f is the strictly increasing and continuous transformation defined in the first paragraph of Section 7.2). Since $f^{-1}(\overline{q}(f(Q(I+1)), s_{I+1}))$ is strictly increasing in s_{I+1} and $\inf S_{I+1} \leq s^*_{I+1}$, the conclusion now follows immediately from Theorem 1 in Milgrom and Roberts [27] or Theorem 1 in Villas-Boas [44] (again see Section 7.2 for further details). Clearly, $Q^*(I) = Q^*(I+1)$ cannot hold unless $s^*_{I+1} = \inf S_{I+1}$, hence the aggregate is strictly increasing whenever the entrant does not choose to be inactive. ■

Our third and final result characterizes the comparative statics of individual strategies. Unlike any of our previous results, this theorem directly uses the implicit function theorem. As such it is a *local* result and also requires that the strategies for the equilibrium in question are interior. The idea here is very simple: Once we have established the effect of a change in t on the aggregate Q, we can treat both t and Q as exogenous variables for any player i and address the individual comparative statics effect through standard techniques. Here we use the implicit function theorem, but there is nothing in the way of using more sophisticated methods such as those of Milgrom and Shannon [28].

Theorem 8. ***(Individual Comparative Statics)*** *Let the conditions of Theorem 6 be satisfied and consider player i's equilibrium strategy $s^*_i(t)$ associated with the smallest (or largest) equilibrium aggregate at some equilibrium $s^* = s^*(t)$ given $t \in T$. Assume that the equilibrium s^* lies in the interior of S and that t is a positive shock. Then the following results hold.*

- *$s^*_i(t)$ is (coordinatewise) locally increasing in t provided that*

$$-[D_{s_i}\Psi_i(s^*_i, g(s^*), t)]^{-1} D_Q \Psi_i(s^*_i, g(s^*), t) \geq 0$$

- *Suppose that the shock t does not directly affect player i (i.e., $\pi_i = \pi_i(s)$). Then the sign of each element of the vector $D_t s^*_i(t)$ is equal to the sign of each element of the vector $-[D_{s_i}\Psi_i(s^*_i, g(s^*))]^{-1} D_Q \Psi_i(s^*_i, g(s^*))$. In particular, $s^*_i(t)$ will be (coordinatewise) locally decreasing in t whenever:*

$$-[D_{s_i}\Psi_i(s^*_i, g(s^*))]^{-1} D_Q \Psi_i(s^*_i, g(s^*)) \leq 0$$

Proof. By the implicit function theorem, we have:

$$D_{s_i}\Psi_i(s_i, Q, t)ds_i = -D_Q\Psi_i(s_i, Q, t)dQ - D^2_{s_i t}\Pi_i(s_i, Q, t)dt.$$

$-[D_{s_i}\Psi_i(s_i,Q,t)]^{-1}D^2_{s_it}\Pi_i(s_i,Q,t)$ is equal to $D_t b_i(Q,t)$ where b_i is player i's backward reply function. From Lemma 5 (the proof) this matrix it non-negative. The results therefore follow directly from the fact that Q increases with t when Q is either the smallest or largest equilibrium aggregate. ■

An application of this result to contests is given in Section 5.2.

4.1. Further Remarks and Extensions

4.1.1. Alternatives to Local Solvability

We can dispense with the differentiability requirements of nice games, and at the same time weaken the assumed pseudo-concavity to quasi-concavity. The resulting conditions are interesting both for applications and because they allow us to discuss the literature's main results on uniqueness of equilibrium in aggregative games (Corchón [7] and Cornes and Hartley [9]).

Let us simplify the exposition by focusing on one-dimensional strategy sets. Recall that an aggregator always has a representation of the form $g(s) = H(\sum_{j=1}^{I} h_j(s_j))$, where H and $h_1,\ldots,h_i$ are strictly increasing functions. Therefore, for any Q in the range of g, we have $Q = g(s) \Leftrightarrow s_i = h_i^{-1}\left[H^{-1}(Q) - \sum_{j\neq i} h_j(s_j)\right]$. Intuitively, this means that if we know the aggregate Q and the aggregate of $I-1$ players $\sum_{j\neq i} h_j(s_j)$, we also know the strategy of the last player s_i. Define a function $G_i(Q,y) \equiv h_i^{-1}[H^{-1}(Q) - y]$ that captures this feature of an aggregative game. Recall from Milgrom and Shannon [28] that a function $f(Q,y)$ satisfies the *single-crossing property* in (Q,y) if, for all $Q' > Q$ and $y' > y$, we have

$$f(Q',y) \geq (>)\ f(Q,y) \Rightarrow f(Q',y') \geq (>)\ f(Q,y').$$

Consider now an aggregative game that satisfies the general compactness and continuity conditions presented at the beginning of Section 2, and in addition has convex strategy sets and payoff functions that are quasi-concave in own strategies. It can then be shown that if $\Pi_i(G_i(Q,y),Q,t)$ satisfies the single-crossing property in (Q,y) for each $i \in \mathcal{I}$, then the conclusions of Theorems 6 and 7 continue to hold. When payoff functions are twice differentiable and the equilibrium is interior, the conclusions of Theorem 8 also carry over.[19]

The previous observations provides a useful and simple alternative to the local solvability condition.

Theorem 9. *Consider a nice aggregative game with linear aggregator $g(s) = \sum_i s_i$ and one-dimensional strategy sets, and assume that for each player $i \in \mathcal{I}$:*

$$D_{s_i}\Psi_i(s_i,Q,t) \leq 0 \text{ for all } s_i \in S_i,\ Q \in X, \text{ and } t \in T \quad . \tag{13}$$

Then the conclusions of Theorems 6, 7, and 8 hold (without any boundary conditions).

Proof. Since g is linear, $G_i(Q,y) = Q - y$ and $\Pi_i(G_i(Q,y),Q,t) = \Pi_i(Q-y,Q,t)$. The condition $D_{s_i}\Psi_i(s_i,Q) \leq 0$ is equivalent to $-D_{s_i}\Psi_i = -D^2_{11}\Pi_i - D^2_{21}\Pi_i \geq 0$ for all s_i and Q. This is in turn equivalent to $\Pi_i(Q-y,Q,t)$ exhibiting increasing differences in Q and y. Since increasing differences implies the single-crossing property, the result now follows from the previous observations. ■

Note that (13) is neither weaker or stronger than the local solvability condition which requires that $D_{s_i}\Psi_i(s_i,Q,t) \neq 0$ for all s_i, Q, and t with $\Psi_i(s_i,Q,t) = 0$.[20] If (13) holds with *strict* inequality throughout, *i.e.*, if,

$$D_{s_i}\Psi_i(s_i,Q,t) < 0 \text{ for all } s_i \in S,\ Q \in X, \text{ and } t \in T \quad , \tag{14}$$

[19] For detailed proofs of these claims, see the working paper version of this paper (Acemoglu and Jensen [2]).

[20] Also note that Theorem 9 is valid even when the game is not nice as explained previously. In particular, pseudo-concavity of payoff functions in own strategies may be replaced with quasi-concavity of payoff functions in own strategies.

then local solvability *is* implied in nice games however (in fact this implies *uniform* local solvability). What Corchón [7] calls "strong concavity" is condition (14) together with the strict inequality version of the strategic substitutes condition (5) of Section 3 (usually called *strict* strategic substitutes). "Strong concavity" is of course stronger than anything assumed in this paper, in particular, it implies uniqueness of equilibrium (Corchón [7], p.156).[21]

4.1.2. Ordinality

It is useful to note that the local solvability condition is *ordinal*. *Firstly*, it is easily seen to be independent of strictly increasing transformations of the payoff functions, *i.e.*, the local solvability holds for the payoff function $\pi_i(s,t)$ if and only if it holds for $\Phi_i(\pi_i(s,t))$ where $\Phi_i : \mathbb{R} \to \mathbb{R}$ is any strictly increasing and twice continuously differentiable function, with derivative denoted by Φ_i' (where differentiability is needed here to ensure that the transformed payoff function is also twice continuously differentiable). *Secondly*, local solvability holds for any choice of coordinate system allowing one to replace each strategy vector s_i with a transformed vector $\tilde{s}_i = \psi_i(s_i)$ where $\psi_i : \mathbb{R}^N \to \mathbb{R}^N$ is a diffeomorphism.[22] *Finally*, if local solvability holds for one aggregator g then it holds also for any aggregator that is a strictly increasing transformation of g. The straight-forward verifications of this last claim is left to the reader.

Ordinality is important in understanding the condition's content, for checking the local solvability condition in certain applications, and plays a critical role in the proof of Theorem 6.

5. Applying the Theorems

In this section, we study a number of applications and show that our methods allow very general comparative static results in these widely used models. We begin with the public good provision model which, as we show is a game of strategic substitutes if the private good is normal, and satisfies the uniform local solvability condition if the public good is strictly normal. If the public good is merely assumed to be normal, the results from Section 4.1.1 apply. These observations allow us to illustrate both our results on games with strategic substitutes and nice games under (uniform) local solvability. We then turn to contests which feature neither strategic substitutes nor strategic complementarities, but, as we show, satisfy the uniform local solvability condition if players' cost functions have larger curvature than their contest success functions. Our third application is the Cournot model where we focus on the case with strategic substitutes and present what we believe are the literature's first comparative statics results at the level of generality of Novshek [30] and Kukushkin [22] — in particular, the results are valid without any quasi-concavity assumptions on firms' profit functions thus allowing for non-decreasing returns to scale. Finally, we illustrate how our results can be applied to games with multidimensional strategies by looking at a model of technology choice in oligopolistic competition.

5.1. Private Provision of Public Goods

Consider the workhorse model of public good provision originally studied by Bergstrom et al. [6]. There are I individuals, each making a voluntary contribution to the provision of a unique public good. Individual

[21] An alternative, and weaker, set of conditions that imply uniqueness are the uniform local solvability condition together with the following condition:

$$\Psi_i(s_i,Q,t)=0 \Rightarrow s_i D_{s_i}\Psi_i(s_i,Q,t)+QD_Q\Psi_i(s_i,Q,t)<0 \text{ for all } s_i,Q, \text{ and } t$$

As explained by Cornes and Hartley [9], these conditions together imply that *share functions* are decreasing which in turn implies that the equilibrium aggregate must be unique. For details see Section 9.1. in Cornes and Hartley [9].

[22] In the new coordinate system, the local solvability condition reads:

$$D\psi_i^{-1}(\tilde{s}_i)\Psi_i(\psi_i^{-1}(\tilde{s}_i),Q)=0 \Rightarrow |D_{\tilde{s}_i}[D\psi_i^{-1}(\tilde{s}_i)\Psi_i(\psi_i^{-1}(\tilde{s}_i),Q)]| \neq 0 \tag{15}$$

But since $D\psi_i^{-1}(\tilde{s}_i)$ is a full rank matrix, (i) $\Psi_i(\psi_i^{-1}(\tilde{s}_i),Q) = 0 \Leftrightarrow D\psi_i^{-1}(\tilde{s}_i)\Psi_i(\psi_i^{-1}(\tilde{s}_i),Q) = 0$; and (ii) $|D_{s_i}\Psi_i(\psi_i^{-1}(\tilde{s}_i),Q)| \neq 0 \Leftrightarrow |D_{\tilde{s}_i}[D\psi_i^{-1}(\tilde{s}_i)\Psi_i(\psi_i^{-1}(\tilde{s}_i),Q)]| = |D\psi_i^{-1}(\tilde{s}_i)D_{s_i}\Psi_i(\psi_i^{-1}(\tilde{s}_i),Q)\ [D\psi_i^{-1}(\tilde{s}_i)]^T| \neq 0$. It follows that the local solvability in the new coordinate system (15) holds if and only if the local solvability condition holds in the original coordinate system.

i maximizes her utility function,

$$u_i(c_i, \sum_{j=1}^{I} s_j + \bar{s}) \ , \tag{16}$$

subject to the budget constraint $c_i + ps_i = m_i$. Here $m_i > 0$ is income, c_i private consumption, and s_i is agent i's contribution to the public good, so that $\sum_{j=1}^{I} s_j + \bar{s}$ is total amount of the public good provided. The exogenous variable $\bar{s} \geq 0$ can be thought of as the state's baseline provision of the public good that will be supplied without any private contributions.

Substituting for c_i, this is seen to be an aggregative game with reduced payoff functions given by

$$\Pi_i \left(s_i, \sum_{j=1}^{I} s_j, m, p, \bar{s} \right) = u_i \left(m_i - ps_i, \sum_{j=1}^{I} s_j + \bar{s} \right) \ , \text{for all } i \in \mathcal{I}. \tag{17}$$

The aggregator is simply $g(s) = \sum_{j=1}^{I} s_j$.[23] When $s^* = (s_i^*)_{i \in \mathcal{I}}$ is an equilibrium, we refer to $g(s^*) = \sum_{i=1}^{I} s_i^*$ as the *aggregate equilibrium provision*. Let us simplify the exposition and notation here by assuming that u_i is smooth and that strategy sets are intervals of the type $S_i = [0, \bar{s}_i] \subseteq \mathbb{R}$. The private good is *normal* if the following condition holds for all $s \in S$:[24]

$$-pD_{12}^2 u_i \left(m_i - ps_i, \sum_{j=1}^{I} s_j + \bar{s} \right) + D_{22}^2 u_i \left(m_i - ps_i, \sum_{j=1}^{I} s_j + \bar{s} \right) \leq 0. \tag{18}$$

Since the left-hand side of (18) is equal to $D_{s_i s_j}^2 \pi_i$, the private good is normal *if and only if* the game has strategic substitutes (cf. Definition 3). The following results consequently follow directly from those in Section 3:

Proposition 1. *Consider the public good provision game and assume that the private good is normal. Then there exists an equilibrium. Furthermore:*

1. *An increase in the state's baseline provision $\bar{s}$ leads to a decrease in the smallest and largest aggregate equilibrium provisions.*
2. *The entry of an additional agent leads to a decrease in the smallest and largest aggregate equilibrium provisions by existing agents.*
3. *A positive shock to agent i will lead to an increase in that agent's smallest and largest equilibrium provisions and to a decrease in the associated aggregate provisions of the remaining $I - 1$ players.*

The observation that the public good provision model has a pure strategy Nash equilibrium assuming merely that the private good is normal appears to be new. The absence of any concavity assumptions highlights that the results of Proposition 1 could not have been derived using the implicit function theorem.[25]

[23] Cornes and Hartley [10] study a more general class of public good provision models where the aggregator is not necessarily linear. As their analysis demonstrates, the resulting model is (still) an aggregative game, and it is therefore a straightforward exercise to extend the result below to it, which we do not do to save on notation.

[24] One way to verify this normality condition is by means of Topkis' theorem: If we substitute for s_i in (16) instead of for c_i, (18) is precisely the condition for the objective to exhibit increasing differences in c_i and m_i. Note that weaker but less standard conditions for normality are available. For example, we could instead of increasing differences in c_i and m_i impose the single-crossing property in c_i and m_i (see footnote 9). Crucially, a normal private good is equivalent to descending best-response correspondences regardless of which normality condition we settle on (again see footnote 9).

[25] This statement also applies to Corchón [7], whose comparative statics results on games with strategic substitutes are indeed based on the implicit function theorem. But even ignoring this, it is easy to see that Corchon's "strong concavity assumption" amounts to assuming that both the private and public goods are strictly normal. This "double normality" assumption (as it is often called) dates all the way back to the original article of Bergstrom et al. [6] and is also in force in Cornes and Hartley [10] mentioned above.

If instead we assume that the public good is strictly normal, we can obtain a number of results using the theorems from Section 4. Indeed, suppose that the payoff function is pseudo-concave (which was not assumed for Proposition 1). Then the public good will be strictly normal if the following condition holds for all $s \in S$:[26]

$$D_{s_i}\Psi_i(s_i, Q) = p^2 D_{11}u_i(m_i - ps_i, Q) - pD_{21}u_i(m_i - ps_i, Q) < 0 \tag{19}$$

It is clear that (19) implies the uniform local solvability condition. In addition, it implies that an increase in m_i or a decrease in p constitute positive shocks, *i.e.*, $D^2_{s_i m}\Pi_i \geq 0$ and $D^2_{s_i p}\Pi_i \leq 0$, respectively. The next proposition therefore follows immediately from Theorems 6-8:

Proposition 2. *Consider the public good provision game and assume that the public good is strictly normal, that payoff functions are pseudo-concave in own strategies and that strategy sets are convex. Then there exists an equilibrium. Furthermore:*

1. *Any positive shock to one or more of the agents (e.g., a decrease in p, or increases in one or more income levels, $m_1, \ldots, m_I$) leads to an increase in the smallest and largest aggregate equilibrium provisions.*
2. *The smallest and largest aggregate equilibrium provisions are increasing in the number of agents.*
3. *The changes in 1 and 2 above are associated with an increase in the provision of agent i if the private good is inferior for this agent, and with a decrease in agent i's provision if the private good is normal and the shock does not directly affect the agent.*

Proposition 2 could also be obtained under weaker conditions by applying Theorem 9. Specifically, if the public good is normal (condition (19) holding as weak inequality), the conditions of that theorem are satisfied and Proposition 2 remains valid. Note also that if one imposes strict normality of the private and public goods simultaneously, then the equilibrium will be unique as proved by Bergstrom et al. [6].

5.2. Models of Contests and Fighting

Consider a contest where I agents are competing to obtain the prize (or fighting for victory). Agent $i \in \mathcal{I}$'s payoff function is,

$$\pi_i(s_i, s_{-i}) = V_i \cdot \frac{h_i(s_i)}{R + H\left(\sum_{j=1}^{I} h_j(s_j)\right)} - c_i(s_i), \tag{20}$$

where s_i denotes agent i's *effort*, $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ his cost function, and $V_i > 0$ his valuation of the prize. The contest success functions $h_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \mathcal{I}$, together with $H : \mathbb{R}_+ \to \mathbb{R}_+$ and the parameter $R > 0$ specify the likelihood of winning the prize. Throughout, all functions are assumed to be strictly increasing and twice continuously differentiable. In addition, strategy sets are assumed to be compact intervals, $S_i = [0, \bar{s}_i]$ for all i. The formulation chosen here is fairly general, and allows not just for standard contests (where often R is taken equal to zero), but also includes models of rent-seeking such as Dixit [12] and Skaperdas [38], as well as patent races in the spirit of Loury [23]. It is clear that this is an aggregative game with the aggregator $g(s) = H(\sum_{j=1}^{I} h_j(s_j))$.

Contests generally feature neither strategic substitutes nor strategic complements. Therefore, the results in Section 3 do no apply, nor do any of the well-known results on games with strategic complementarities. In this case, the most obvious strategy for deriving comparative static results is to use the implicit function theorem. Unfortunately, the implicit function theorem approach yields ambiguous conclusions unless one imposes additional, strong assumptions. For this reason, previous treatments have restricted attention to special cases of the above formulation. For example, Tullock [43] studied two-player contests, while Loury [23] focused on symmetric contests under (ad hoc) stability conditions. The most general comparative statics

[26] The equivalence between strict normality of the public good and (19) follows since $\partial s_i(m, p, \sum_{j \neq i} s_j)/\partial m = \alpha(pD^2_{12}u_i - p^2 D^2_{11}u_i)$ where $\alpha > 0$ is a constant.

results available in the literature are to our knowledge those of Nti [29] whose results apply to symmetric equilibria (in particular, agents must be identical) under the additional assumptions that $H = id$ (the identity function), that $h_i = h$ for all i and concave (a symmetric, concave contest success function), and that costs are linear ($c_i(s_i) = \bar{c}s_i$ for some constant $\bar{c} > 0$).

Using the results of Section 4, we can establish considerably more general and robust results on this important class of models. In particular, no symmetry assumptions are imposed what so ever.[27] To apply the results of Section 4, we must verify the local solvability condition. Direct calculations yield

$$\Psi_i(s_i, Q) = V_i \cdot \left[\frac{h_i'(s_i)}{R+Q} - \frac{H'\left(H^{-1}(Q)\right) h_i'(s_i) h_i(s_i)}{(R+Q)^2} \right] - c_i'(s_i),$$

and

$$D_{s_i}\Psi_i(s_i, Q) = \frac{V_i h_i''(s_i)}{h_i'(s_i)} \cdot \left[\frac{h_i'(s_i)}{R+Q} - \frac{H'\left(H^{-1}(Q)\right) h_i'(s_i) h_i(s_i)}{(R+Q)^2} \right] - c_i''(s_i) - V_i \frac{H'\left(H^{-1}(Q)\right) (h_i'(s_i))^2}{(R+Q)^2}.$$

Therefore, when $\Psi_i(s_i, Q) = 0$, we have

$$D_{s_i}\Psi_i = \frac{h_i''(s_i)}{h_i'(s_i)} c_i'(s_i) - c_i''(s_i) - V_i \cdot \frac{H'\left(H^{-1}(Q)\right) (h_i'(s_i))^2}{(R+Q)^2}.$$

Dividing both sides by $c_i'(s_i) > 0$, we immediately see that $D_{s_i}\Psi_i < 0$ whenever $\Psi_i(s_i, Q) = 0$ and the following condition is satisfied:

$$\frac{h_i''(s_i)}{h_i'(s_i)} \leq \frac{c_i''(s_i)}{c_i'(s_i)} \text{ for all } s_i \in S_i. \tag{21}$$

When $s_i > 0$, this can also be written, $\frac{h_i''(s_i)s_i}{h_i'(s_i)} \leq \frac{c_i''(s_i)s_i}{c_i'(s_i)}$ for all $s_i \in S_i$, which says that the cost function must have a larger *curvature* than the contest success function. Intuitively, this simply means that the cost function is "more convex" than the success function (note that this statement does not imply that either function must be convex!). Parallel curvature conditions play a central role in industrial organization, *e.g.*, in the analysis of price discrimination (Schmalensee [34]).

Note that since $D_{s_i}\Psi_i < 0$, condition (21) implies the uniform local solvability condition. Hence the conclusions of Theorem 6 are valid without any boundary conditions on payoff functions. *1.* and *2.* of the following proposition now follow directly from Theorems 6-7. Part *3.* of this proposition is a straightforward application of Theorem 8 (the algebraically cumbersome details are placed in an appendix along with a verification of the existence claim).

Proposition 3. *Consider a contest with payoff functions* (20) *and suppose that H is strictly increasing and convex, h_i and c_i are strictly increasing, that all of these functions are twice continuously differentiable, and that condition* (21) *is satisfied for each agent. Then there exists an equilibrium. Furthermore:*

1. *The smallest and largest aggregate equilibrium efforts are increasing in any positive shock (e.g., a decrease in R or an increase in V_i for one or more players).*
2. *Entry of an additional player increases the aggregate equilibrium effort.*

[27] Since we do not assume concavity of payoff functions, the following proposition also generalizes the existence result of Szidarovszky and Okuguchi [35]. Note that when $R = 0$, $H(0) = 0$, and $h_i(0) = 0$ for all i (as assumed by Szidarovszky and Okuguchi [35]), the payoff functions are not well-defined when all agents choose zero effort. This poses a minor difficulty for the proof of existence, but it can be easily overcome: Simply consider a sequence of games all of which have $R_n > 0$ but are otherwise identical, and use the result below. Letting $R_n \to 0$ we get for each n an equilibrium, and by the upper hemi-continuity of the best-reply correspondences, the limit point of any convergent subsequence of equilibria will then be an equilibrium for the game with $R = 0$.

3. *Finally, the effects on the individual agents' effort levels can be predicted as follows. Define the real-valued function* $\eta : \mathbb{R} \to \mathbb{R}$*:*

$$\eta(Q^*) \equiv \left[\frac{2H'\left(H^{-1}(Q^*)\right)}{(R+Q^*)} - \frac{H''\left(H^{-1}(Q^*)\right)}{H'\left(H^{-1}(Q^*)\right)} \right]^{-1}.$$

Then the changes in parts 1 or 2 above are associated with an increase in the effort of player $i \in \mathcal{I}$ *at the corresponding equilibrium aggregate* Q^* *whenever player* i *is "dominant" in the sense that* $h_i(s_i^*) \geq \eta(Q^*)$*. Conversely, if* i *is "not dominant", i.e.,* $h_i(s_i^*) < \eta(Q^*)$*, then the changes in parts 1 and 2 decrease player* i*'s effort provided that the shock does not affect this player directly (e.g., corresponding to a decrease in another player's costs).*

Proof. *1.* and *2.* were verified above. For the remaining statements see Section 7.3 in the Appendix. ■

Note that when $H = h_i = \text{id}$ (the identity function), and $R = 0$, we have $\eta(Q^*) = Q^*/2$, and so player i is "dominant" in the sense of *3.* if and only if $s_i^* \geq Q^*/2$. In a two-player contest, this precisely means that she is the "favorite" to win the prize in the sense of Dixit [12]. With $I > 2$ players, being "dominant" means that she is more likely to win the prize than everybody else *combined*.

Also observe that the conditions of Proposition 3 are satisfied if H is the identity function, c_i is convex, and h_i is concave.[28] Proposition 3 also covers important cases where h_i is not concave. For example, Hirshleifer [17] proposes the *logit specification* of the contest success function, with $H = id$ (the identity function), and $h_i(s_i) = e^{k_i s_i}$ $(k_i > 0)$, and studies the special case where $k_i = k$ for all i under additional assumptions. In such cases, Proposition 3 continue to apply as long as c_i has larger curvature than h_i at all $s_i \in S_i$. For example if $c_i(s_i) = e^{l_i s_i}$ where $l_i \geq k_i$, this will be the case.

5.3. Cournot Oligopoly

Consider the Cournot model of quantity competition. There are I firms, each choosing $s_i \in [0, \bar{s}_i]$ to maximize profits:

$$\pi_i(s,t) = s_i P\left(\sum_{j=1}^{I} s_j + \bar{t}\right) - c_i(s_i, t_i). \tag{22}$$

Here t_i is a parameter that affects the cost c_i of firm i, and $\bar{t}$ parameterizes shifts in direct demand ($Q = D(p) - \bar{t} \Leftrightarrow p = P(Q + \bar{t})$, where D is the direct demand function and P the indirect demand function). It is clear that increases in $\bar{t}$ (downward shifts in direct demand) are shocks that hit the aggregator in the sense of Definition 4. We assume throughout that $D^2_{s_i t_i} c_i \leq 0$, *i.e.*, that an increase in t_i is a positive shock. Clearly, this game is aggregative with $g(s) = \sum_j s_j$. Moreover, it features strategic substitutes provided that

$$P'(Q + \bar{t}) + s_i P''(Q + \bar{t}) \leq 0, \tag{23}$$

where $Q \equiv \sum_{j=1}^{I} s_j$. For example, this will hold if inverse demand is strictly decreasing and the elasticity of P', $\varepsilon_P(Q) = P''(Q)\, Q / P'(Q)$ is less than 1 (naturally, concave inverse demand is in turn sufficient for this).[29] Note that this condition is completely independent of the cost function, hence the results to follow are valid if firms' production technologies exhibit non-decreasing returns to scale. It is also not required that strategy sets are convex.[30] As in the previous applications, the following follows straight from our results in Section 3:

[28] Szidarovszky and Okuguchi [35] prove that these conditions imply uniqueness of equilibrium if in addition $R = 0$ in (20), see also Cornes and Hartley [8] for a very simple proof of this result based on the "share function approach" discussed in the Introduction and elsewhere. Such uniqueness is not necessary or assumed in Proposition 3, but if it holds one of course gets "sharper" conclusions that refer to the unique equilibrium aggregate.

[29] Amir [4] studies conditions under which the Cournot model will be a game of strategic substitutes or complements (our results on strategic substitutes are equally valid under the ordinal conditions of Milgrom and Shannon [28] which is what Amir focuses on).

[30] To the best of hour knowledge, there are no existing comparative statics results for the Cournot model that hold at this level of generality.

Proposition 4. *Consider the Cournot model and assume that* (23) *holds. Then this is a game with strategic substitutes and the following comparative statics results apply:*

1. *An increase in $\bar{t}$ leads to a decrease in the smallest and largest aggregate equilibrium outputs.*
2. *The entry of an additional firm leads to a decrease in the smallest and largest aggregate equilibrium outputs of the existing firms.*
3. *A positive shock to firm i (an increase in t_i) will lead to an increase in that firm's smallest and largest equilibrium outputs and to a decrease in the associated aggregate equilibrium outputs of the remaining $I-1$ firms.*

Notice that part *3.* directly generalizes the duopoly result of Vives [46] to any number of firms (in the duopoly case, one can reverse the order on one of the firm's strategy sets and obtain a game with strategic complementaritites).

If instead we were to assume concavity (or pseudo-concavity), comparative statics can be obtained by use of the results from Section 4. In the absence of strategic substitutes (*e.g.*, Amir [4]), the comparative statics results one obtains in this way will be new. We omit the details, but see the paragraph after Definition 8 for conditions under which the local solvability condition holds in the Cournot model.

5.4. Technology Choice in Oligopoly

As a final application, we consider games in which oligopoly producers make technology choices as well as setting output. Thus strategy sets are of dimension 2, which allows us to illustrate how our results can be applied when strategy sets are multidimensional. For a general and related discussion of models of technological choice and competition see Vives [48].

Consider a Cournot model with I heterogeneous firms. Let $q=(q_1,...,q_I)$ be the output vector and $a=(a_1,...,a_I)$ the technology vector. Throughout, both are assumed to lie in compact sets. Let us define $Q=\sum_{j=1}^{I} q_j$ as aggregate output. Profit of firm i is

$$\Pi_i(q_i,a_i,Q)\equiv\pi_i(q,a)=q_iP(Q)-c_i(q_i,a_i)-C_i(a_i),$$

where P is the (decreasing) inverse market demand function, the cost function c_i is a function of firm i's quantity and technology choices, and C_i is the cost of technology adoption. Assume that P, c_i and C_i are twice differentiable for all i, P is strictly decreasing ($P'(Q)<0$ for all Q), C_i is convex, and $\partial c_i(q_i,a_i)/\partial q_i\partial a_i<0$ (for each i), so that greater technology investments reduce the marginal cost of production for each firm.

The first-order necessary conditions for profit maximization are

$$\begin{aligned}\frac{\partial\pi_i}{\partial q_i} &= P'(Q)q_i+P(Q)-\frac{\partial c_i(q_i,a_i)}{\partial q_i}=0\\ \frac{\partial\pi_i}{\partial a_i} &= -\frac{\partial c_i(q_i,a_i)}{\partial a_i}-\frac{\partial C_i(a_i)}{\partial a_i}=0.\end{aligned}$$

We assume that these first-order conditions hold at any optimum (including optima at the boundary, cf. part *3.* of Definition 6). Let us now consider the effect of a decline in the cost of technology investment by one of the firms (*i.e.,* a shift in C_i), which clearly corresponds to a positive shock. The results from Section 4 suggest that we should check the local solvability condition. In particular, consider the matrix:

$$D_{(q_i,a_i)}\Psi_i=\begin{pmatrix} P'(Q)-\frac{\partial^2 c_i}{\partial q_i^2} & -\frac{\partial^2 c_i}{\partial q_i\partial a_i}\\ -\frac{\partial^2 c_i}{\partial q_i\partial a_i} & -\frac{\partial^2 c_i}{\partial a_i^2}-\frac{\partial^2 C_i}{\partial a_i^2}\end{pmatrix}$$

for each i, where in terms of the notation of Section 4 we have taken $s_i=(q_i,a_i)$. When $c_i(q_i,a_i)$ is convex, the matrix

$$\begin{pmatrix} -\frac{\partial^2 c_i}{\partial q_i^2} & -\frac{\partial^2 c_i}{\partial q_i\partial a_i}\\ -\frac{\partial^2 c_i}{\partial q_i\partial a_i} & -\frac{\partial^2 c_i}{\partial a_i^2}\end{pmatrix}$$

is negative semi-definite. Since $P'(Q) < 0$ and $-\partial^2 C_i/\partial a_i^2 \leq 0$, this is sufficient to guarantee that $|D\Psi_i| < 0$. Therefore, whenever each $c_i(q_i, a_i)$ is convex, the local solvability condition is satisfied. Hence, a decline in the cost of technology investments for one of the firms will necessarily increase total output. Similarly, the effects of an increase in demand on output and technology choices can be determined robustly. The following proposition summarizes these results (proof omitted):

Proposition 5. *Consider the technology adoption game described above and assume that the cost functions $c_i = c_i(q_i, a_i)$ (for each i) are convex. Then the local solvability condition holds and as a consequence:*

1. *Any positive shock to one or more of the firms (e.g., a decrease in marginal costs parameterized via $c_i = c_i(q_i, a_i, t)$) will lead to an increase in total equilibrium output.*
2. *Entry of an additional firm will lead to an increase in total output.*

Note also that oligopoly-technology game is a game with strategic substitutes when $\partial^2 c_i(q_i, a_i)/\partial q_i \partial a_i \leq 0$.[31] So when technological development lowers the marginal cost of producing more input, the results from Section 3 will apply and produce a parallel set of comparative statics results.

6. Conclusion

This paper presented robust comparative static results for aggregative games and showed how these results can be applied in several diverse settings. In aggregative games, each player's payoff depends on her own actions and on an aggregate of the actions of all players (for example, sum, product or some moment of the distribution of actions). Many common games in industrial organization, political economy, public economics, and macroeconomics can be cast as aggregative games. Our results focused on the effects of changes in various parameters on the aggregates of the game. In most of these situations the behavior of the aggregate is of interest both directly and also indirectly, because the comparative statics of the actions of each player can be obtained as a function of the aggregate. For example, in the context of a Cournot model, our results characterize the behavior of aggregate output, and given the response of the aggregate to a shock, one can then characterize the response of the output of each firm in the industry.

We focused on two classes of aggregative games: (1) aggregative games with strategic substitutes and (2) nice aggregative games, where payoff functions are twice continuously differentiable, and (pseudo-)concave in own strategies. For instance, for aggregative games with strategic substitutes, we showed that:

1. A change in a parameter that directly affects the aggregate — such as a negative shock to demand in the Cournot model — will lead to a decrease in the aggregate (in the sense that the smallest and the largest elements of the set of equilibrium aggregates increase).
2. Entry of an additional player decreases the (appropriately defined) aggregate of the existing players.
3. A "positive" idiosyncratic shock, defined as a parameter change that increases the marginal payoff of a single player, leads to an increase in that player's strategy and a decrease in the aggregate of other players' strategies.

We also provided parallel results for nice games under a condition called the *local solvability condition*. Those results apply to for example contests which are not games of strategic substitutes (nor are they games of strategic complementarities).

The framework developed in this paper can be applied to a variety of settings to obtain "robust" comparative static results that hold without specific parametric assumptions. In such applications, our approach often allows considerable strengthening of existing results and also clarifies the role of various assumptions used in previous analyses. We illustrated how these results can be applied and yield sharp results using several examples, including public good provision games, contests, and oligopoly games with technology choice.

[31] This condition ensures that payoff functions are supermodular in own strategies. It is easy to check that payoff functions also exhibit decreasing differences in own and opponents' strategies.

Throughout this paper, the aggregate has been assumed to be *one-dimensional*. It is possible to extend the framework to allow for multi-dimensional aggregates (Acemoglu and Jensen [2]). Other restrictions of this paper are that strategy sets are assumed to be *finite dimensional* and the set of players *finite*. We conjecture that the results presented in this paper can be generalized both to games with infinitely many players and games with infinite-dimensional strategy sets. Indeed, with the appropriate definition of an aggregator for a game with infinitely many players (*e.g.,* along the lines of the separability definitions in Vind and Grodal [45], Chapters 12-13), our main results and in fact even our proofs remain valid in this case. Similarly, with the appropriate local solvability condition, all of our results on nice games appear to generalize to games with infinite-dimensional strategy sets.

7. Appendix

7.1. Details of the Novshek Selection from the Proof of Theorem 2

In this section we give a detailed exposition of the *Novshek selection* from the proof of Theorem 2. Note that the construction here is slightly different from the original one in Novshek [30], but the basic intuition is the same. Aside from being somewhat briefer, the present way of constructing the "Novshek selection" does not suffer from the "countability problem" in Novshek's proof pointed out by Kukushkin [22], since we use Zorn's Lemma to construct the selection.

Definition 10. ***(Novshek Selections)*** *Let $Q^a, Q^b \in \mathbb{R}$, $Q^a \leq Q^b$. A selection $q : [Q^a, Q^b] \to \mathbb{R}$ from Z (i.e., a function with $q(Q,t) \in Z(Q,t)$ for all $Q \in [Q^a, Q^b]$) is called a Novshek selection (on $[Q^a, Q^b]$) if the following hold for all $Q \in [Q^a, Q^b]$:*

1. *$q(Q,t) \geq z$ for all $z \in Z(Q,t)$.*
2. *$q(Q,t) \leq Q$.*
3. *The backward reply selections $b_i(Q,t) \in \tilde{B}_i(Q,t)$ associated with q (i.e., backward reply selections satisfying $q(Q,t) = \sum_j b_j(Q,t)$ all Q) are all decreasing in Q on $[Q^a, Q^b]$,* i.e., *$Q'' \geq Q' \Rightarrow b_i(Q'',t) \leq b_i(Q',t)$.*

Before we can construct a suitable Novshek selection, we need to establish the existence of an element $Q^{\max} > 0$ as in Figure 1, with the property that $q < Q^{\max}$ for all $q \in Z(Q^{\max}, t)$. This can be done by suitably modifying an argument of Kukushkin [22] (p. 24, l.18-20).

Lemma 1. *There exists an element $Q^{\max} > 0$ such that $q < Q^{\max}$ for all $q \in Z(Q^{\max}, t)$.*

Proof. Let D_i denote the subset of $\mathbb{R}$ upon which $h_i \circ \tilde{R}_i$ is defined, *i.e.*, write $\gamma \in D_i$ if and only if $h_i \circ \tilde{R}_i(\gamma) \neq \emptyset$. Since $h_i \circ \tilde{R}_i$ is upper hemi-continuous, D_i is closed. It is also a bounded set since $\tilde{R}_i \subseteq S_i$ and each S_i is compact. Consequently, D_i has a maximum, which we denoted by d_i. Then extend $h_i \circ \tilde{R}_i$ from D_i to $D_i \cup (d_i, Q^{\max}]$ by taking $h_i \circ \tilde{R}_i(d) \equiv \perp_i$ all $d \in (d_i, Q^{\max}]$. Here $\perp_i$ can be any small enough element (for each player $i \in \mathcal{I}$) such that $\sum_i \perp_i < Q^{\max}$, $\perp_i \leq \min h_i \circ \tilde{R}_i(d_i)$, and $Q^{\max} - \perp_i \in (d_i, Q^{\max}]$. With Z defined as above but based on the previous extension of $h_i \circ R_i$ to $D_i \cup (d_i, Q^{\max}]$, it is clear that $Z(Q^{\max}, t) = \{\sum_i \perp_i\} < Q^{\max}$ which is what we wanted to show. ■

Note that, strictly speaking, Z in the previous lemma refers to the aggregate backward reply correspondence *after* best-response correspondences have been extended as in the proof. In particular, therefore $Z(Q^{\max}, t) \neq \emptyset$. Let $D \subseteq (-\infty, Q^{\max}]$ denote the subset of $\mathbb{R}$ upon which (the extended version of) $Z(\cdot, t)$ is well-defined. Abusing notation slightly, let $[Q', Q^{\max}] \equiv D \cap \{Q : Q' \leq Q \leq Q^{\max}\}$. Any such interval $[Q', Q^{\max}]$ will be compact because D is compact (see the proof of the previous lemma for an identical argument).

Lemma 2. *There exists an element $Q^{\min} \leq Q^{\max}$ and a well-defined Novshek selection $q : [Q^{\min}, Q^{\max}] \to \mathbb{R}$ on $[Q^{\min}, Q^{\max}]$. The element $Q^{\min}$ will be minimal in the sense that if $Q' < Q^{\min}$, then there will not exist a Novshek selection on $[Q', Q^{\max}]$.*

Proof. Denote by $\Omega \subseteq 2^{\mathbb{R}}$ the *set* of all "intervals" $[Q', Q^{\max}]$ upon which a selection with properties 1.-3. exists. Notice that $\{Q^{\max}\} \in \Omega$ so Ω is not empty. Ω is ordered by inclusion since for any two elements ω', ω'' in Ω, $\omega'' = [Q'', Q^{\max}] \subseteq [Q', Q^{\max}] = \omega' \Leftrightarrow Q'' \leq Q'$. A *chain* in Ω is a totally ordered subset (under inclusion). It follows directly from the upper hemi-continuity of the backward reply correspondences that any such chain with an upper bound has a supremum in Ω (*i.e.*, Ω contains an "interval" that contains each "interval" in the chain). Zorn's Lemma therefore implies that Ω contains a maximal element, *i.e.*, there exists an interval $[Q^{\min}, Q^{\max}] \in \Omega$ that is not (properly) contained in any other interval from Ω. ■

7.2. Proof of Theorem 6

We begin by noting that there is no loss of generality in using the aggregator $g(s) \equiv \sum_i h_i(s_i)$ in the following, and assuming that $\min_{s_i \in S_i} h_i(s_i) = 0$ for all i. To see why, recall that the local solvability condition is independent of any strictly increasing transformation of the aggregator as well as any coordinate shift (Section 4.1.2). Let the original aggregator be $\tilde{g}(s) = H(\sum_i \tilde{h}_i(s_i))$. We begin by transforming strategy vectors by multiplying with a positive constant such that $\max_{s_i \in S_i} \tilde{h}_i(s_i) - \min_{s_i \in S_i} \tilde{h}_i(s_i) = 1$. Next, we use the transformation $f(z) = H^{-1}(z) - \sum_i \min_{s_i \in S_i} \tilde{h}_i(s_i)$ to get the new aggregator $g(s) \equiv f(\tilde{g}(s)) = \sum_i h_i(s_i)$, where $h_i(s_i) \equiv \tilde{h}_i(s_i) - \min_{s_i \in S_i} \tilde{h}_i(s_i)$. Clearly, $\min_{s_i \in S_i} h_i(s_i) = 0$ for all i with this transformed aggregator.

Let $R_i : S_{-i} \times T \to S_i$ be the best response correspondence of player i and $\tilde{R}_i$ the transformed and reduced best response correspondence defined by $\tilde{R}_i(\sum_{j \neq i} h_j(s_j), t) \equiv h_i \circ R_i(s_{-i}, t)$. Then define the (transformed) *backward reply correspondence* B_i of player i by means of:

$$\eta_i \in B_i(Q, t) \Leftrightarrow \eta_i \in \tilde{R}_i(Q - \eta_i, t)$$

It is clear that Q is an equilibrium aggregate given $t \in T$ if and only if $Q \in Z(Q, t) \equiv \sum_i B_i(Q, t)$ (the correspondence Z is the aggregate backward reply correspondence already studied in the proof of Theorem 2).

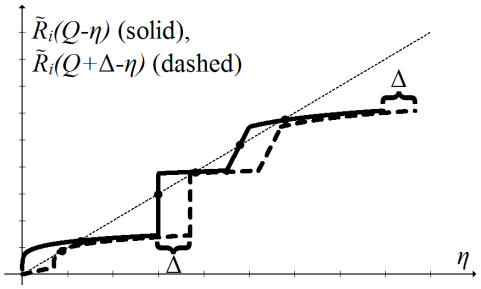

Figure 8: $\eta \in B_i(Q)$ $[\eta \in B_i(Q + \Delta)]$ if and only if the solid [dashed] curve intersects the diagonal at η.

We are going to suppress t to simplify notation in what follows. By definition, $\eta \in B_i(Q) \Leftrightarrow \eta \in \tilde{R}_i(Q - \eta)$. Graphically, η lies in B_i if and only if the correspondence $\tilde{R}_i(Q - \cdot)$ intersects with the diagonal/45°-line at η. A crucial feature of the graphs of $\tilde{R}_i(Q - \cdot)$ for different values of Q, is that these correspond to "horizontal parallel shifts" of each other. To be precise, consider the solid curve in figure 1 which is the graph of $\tilde{R}_i(Q - \cdot)$ for some choice of Q. Now increase Q to $Q + \Delta$, $\Delta > 0$. Because of the *additive* way in which η and Q enter into $\tilde{R}_i$, the graph of $\tilde{R}_i(Q + \Delta - \cdot)$ will precisely be a *parallel right shift* of the graph of $\tilde{R}_i(Q - \cdot)$ with each point on the former laying precisely Δ to the right of each point on the latter (the dashed curve in figure 8). Similarly, if $\Delta < 0$, the graph will be shifted to the *left* in a parallel fashion. It is this straight-forward observation that drives essentially the entire proof. We begin with the following:

Lemma 3. *When $S_i \subseteq \mathbb{R}$ and the uniform local solvability condition holds, we may for each player $i \in \mathcal{I}$ replace Ψ_i with a function $\tilde{\Psi}_i$ such that (i) $\eta_i \in B_i(Q, t)$ if and only if $\tilde{\Psi}_i(s_i, Q, t) = 0$ for some $s_i \in S_i$ with $\eta_i = h_i(s_i)$, and (ii) $D_{s_i}\tilde{\Psi}_i(s_i, Q, t) < 0$ whenever $\tilde{\Psi}_i(s_i, Q, t) = 0$.*

Proof. We suppress t in the following to simplify notation. Obviously, the statement is valid for any η_i in the interior of S_i by assumption. So there is only something to prove when $S_i \subseteq \mathbb{R}$ and $\eta_i' \in B_i(Q') \cap \{0, \max S_i\}$ (*i.e.*, is on the boundary) and $\Psi_i(\eta_i', Q') \neq 0$. Consider here the case where $\eta_i' = 0$ (the proof is the same when $\eta_i' = \max S_i$). Let $[Q_a, Q_b]$ be the maximal interval (necessarily closed) for which $\{0\} \in B_i(Q)$ for all $Q \in [Q_a, Q_b]$. It is easy to show that we must have $\Psi_i(0, Q_a) = \Psi_i(0, Q_b) = 0$: by varying Q either below Q_a [or above Q_b] we get a continuous, non-constant extension $b_i(Q) \in B_i(Q)$ with $b_i(Q_a) = 0$ [$b_i(Q_b) = 0$]. In particular, such an extension must lie in the interior of S_i for $Q \neq Q_a$ [$Q \neq Q_b$]. But then $\Psi_i(b_i(Q), Q) = 0$ for all $Q \neq Q_a$, and by continuity of b_i and Ψ_i follows that $\Psi_i(0, Q_a) = 0$ [$\Psi_i(0, Q_b) = 0$ respectively]. Importantly, by uniform local solvability $\Psi_i(0, Q_a) < 0$ and $\Psi_i(0, Q_b) < 0$. We may therefore replace Ψ_i with a function $\tilde{\Psi}_i$ which equals Ψ_i outside any interval $[Q_a, Q_b]$ and where: (i) $\tilde{\Psi}_i(0, Q) = 0$ for all $Q \in [Q_a, Q_b]$, and (ii) $D_{s_i}\tilde{\Psi}_i(0, Q) < 0$ for all $Q \in [Q_a, Q_b]$. Clearly, we can also choose $\tilde{\Psi}_i$ such that the resulting "replacement" of Ψ_i will be continuously differentiable in s_i. Observe that without uniform local solvability, we might not have been able to find a replacement satisfying (ii) because we could have, say, $D_{s_i}\Psi_i(0, Q_a) < 0$ and $D_{s_i}\Psi_i(0, Q_b) > 0$. ■

Note that the conclusion of lemma 3 is true by assumption if boundary conditions hold. Note also that in the multi-dimensional case where boundary conditions are always in force, the statement remains valid by assumption if we replace *(ii)* with $|D_{s_i}\Psi_i(s_i, Q, t)| \neq 0$ whenever $\Psi_i(s_i, Q, t) = 0$ (this is the local solvability condition). We can now prove a key observation:

Lemma 4. *$B_i(Q)$ consists of at most one element (hence B_i is single-valued whenever it is well-defined).*

Proof. As is clear graphically, if $B_i(Q)$ is not single-valued for some Q, there must lie at least one point (x_i, y_i) on the graph of $\tilde{R}_i(Q - \cdot)$ ((x_i, y_i), $y_i \in \tilde{R}_i(Q - x_i)$) with the property that a line with slope $+1$ intersects the graph precisely at (x_i, y_i) and in a neighborhood either lies entirely below or entirely above the graph. Since $y_i \in \tilde{R}_i(Q + y_i - (x_i + y_i))$, it follows that $x_i + y_i \in B_i(Q + y_i)$. But either raising or lowering Q will now lead to two continuous selections from B_i, call them b_i and $\tilde{b}_i$ which take the same value at $Q' + y_i$ (*i.e.*, $b_i(Q' + y_i) = \tilde{b}_i(Q' + y_i)$) but take different values at all $Q + y_i$ with $Q \neq Q'$ and Q sufficiently close to Q'. We are now going to show that this is impossible under the local solvability condition.

Begin by noting that in the one-dimensional case by lemma 3 (denoting here $\tilde{\Psi}_i$ of that lemma again by Ψ_i), $\eta_i' \in B_i(Q') \Rightarrow [\Psi_i(s_i', Q') = 0$ for some $s_i' \in S_i$ with $h_i(s_i') = \eta_i']$. And since then $D_{s_i}\Psi_i(s_i', Q') < 0$, the implicit function theorem implies the existence of a locally unique, differentiable function $f_i : (Q' - \epsilon, Q' + \epsilon) \to S_i$ such that $\Psi_i(f_i(Q), Q) = 0$ for all $Q \in (Q' - \epsilon, Q' + \epsilon)$, and such that $f_i(Q') = s_i'$. This clearly contradicts the existence of selections b_i and $\tilde{b}_i$ as described above and the proof is complete. In the multidimensional case, the implicit function theorem still applies due to the local solvability condition, so we still get functions b_i and $\tilde{b}_i$ as described. But things are complicated by the fact that we may have two different solutions to $\Psi_i(\cdot, Q') = 0$: $\Psi_i(s_i, Q') = 0$ and $\Psi_i(\tilde{s}_i, Q') = 0$, $s_i \neq \tilde{s}_i$ with $\eta_i' \equiv h_i(s_i) = h_i(\tilde{s}_i)$. Intuitively, the problem here is that local uniqueness in terms of s_i does not (seem!) to imply local uniqueness in terms of $\eta_i = h_i(s_i)$ as stated in the lemma. Since $s_i, \tilde{s}_i \in R_i(Q' - \eta_i')$, the above situation can of course only arise if R_i is not single-valued at $Q' - \eta_i'$. In fact, it can only happen if $\tilde{R}_i$ is not single-valued at $Q' - \eta_i'$ since otherwise $\eta_i = h_i(s_i)$ for all $s_i \in R_i(Q' - \eta_i')$ (a convex set), which definitely contradicts local uniqueness of solutions to $\Psi_i(\cdot, Q') = 0$. Now, when such multiplicity in terms of s_i arises, the implicit function theorem will give us two functions f_i and $\tilde{f}_i$ such that $\Psi_i(f_i(Q), Q) = 0$, $\Psi_i(\tilde{f}_i(Q), Q) = 0$, and $f_i(Q) \neq \tilde{f}_i(Q)$ for Q close to Q' (in addition, $f_i(Q') = s_i$ and $\tilde{f}_i(Q') = \tilde{s}_i$). Since h_i, f_i and $\tilde{f}_i$ are differentiable at Q', $Q - h_i(f_i(Q))$ and $Q - h_i(\tilde{f}_i(Q))$ will obviously be differentiable at Q'. Neither term can be constant in Q: If this were the case for, say, $Q - h_i(f_i(Q))$ we would have $h_i(f_i(Q)), \eta_i' \in \tilde{R}_i(Q - h_i(f_i(Q)) = \tilde{R}_i(Q' - \eta_i')$ where necessarily $h_i(f_i(Q)) \neq h_i(f_i(Q')) = \eta_i'$ (for $Q - h_i(f_i(Q))$ to be locally constant at Q', $f_i(Q)$ obviously cannot be locally constant at Q'). But then we can for any Z close to $Q' - \eta_i'$ find Q and $\tilde{Q}$ such that

$Z = Q - h_i(f_i(Q)) = \tilde{Q} - h_i(\tilde{f}_i(\tilde{Q}))$, and since $f_i(Q), \tilde{f}_i(\tilde{Q}) \in \tilde{R}_i(Z)$, $\tilde{R}_i$ must be multi-valued not just at $Q' - \eta_i'$ but also at any Z close to $Q' - \eta_i'$. This again leads to a contradiction and the proof is complete. ■

In the following, let b_i be the function such that $B_i(Q) = \{b_i(Q)\}$ (of course, $b_i(Q)$ is only well-defined if $B_i(Q) \neq \emptyset$). Let $\theta_i \equiv \max \tilde{R}_i(0)$ and $\rho_i \equiv \min \tilde{R}_i(\overline{x}_i)$ where $\overline{x}_i \equiv \sum_{j \neq i} \max_{s_j \in S_j} h_j(s_j)$. Since $\theta_i \in \tilde{R}_i(\theta_i - \theta_i)$ and $\rho_i \in \tilde{R}_i(\overline{x}_i + \rho_i - \rho_i)$, we must have $b_i(\theta_i) = \theta_i$ and $b_i(\overline{x}_i + \rho_i) = \rho_i$. It can never be the case that $\overline{x}_i + \rho_i = \theta_i$.[32] Hence the previous construction marks two different points on the backward reply function b_i. Assume first that $\theta_i < \overline{x}_i + \rho_i$. Then the graph of $\tilde{R}_i(\overline{x}_i + \rho_i - \eta)$ must lie strictly below the 45° line for all $\eta > \rho_i$ since if not it would lie everywhere above the diagonal, which would imply that $B_i(\theta_i) = \emptyset$ (observe that we are here using that B_i is single-valued since this implies that $\tilde{R}_i(Q - \cdot)$ cannot intersect with the 45°-line twice). Likewise, the graph of $\tilde{R}_i(\theta_i - \eta)$ must lie completely above the 45°-line for $\eta < \theta_i$, otherwise we would have $B_i(\overline{x}_i + \rho_i) = \emptyset$. In case $\theta_i > \overline{x}_i + \rho_i$, the "dual" conclusions apply for the same reasons (by "dual" we mean that $\tilde{R}_i(\overline{x}_i + \rho_i - \eta)$ lies above the 45° line for $\eta > \rho_i$ and $\tilde{R}_i(\theta_i - \eta)$ lies below the 45°-line for $\eta < \theta_i$). From now on we are going to focus on the first of the above cases where $\theta_i < \overline{x}_i + \rho_i$. But all arguments immediately carry over to the case where $\theta_i > \overline{x}_i + \rho_i$ so that b_i will be defined on $[\overline{x}_i + \rho_i, \theta_i]$ instead of $[\theta_i, \overline{x}_i + \rho_i]$ (simply interchange the left and right end-points θ_i and $\overline{x}_i + \rho_i$ throughout the following arguments).

The next three conclusions follow immediately from the fact that a change in Q, from Q to $Q + \Delta$ say, corresponds to an exact parallel shift of the graph of $\tilde{R}_i(Q - \cdot)$ either to the left $\Delta < 0$ or to the right $\Delta > 0$. First, we see that $B_i(Q) = \emptyset$ for all $Q \notin [\theta_i, \overline{x}_i + \rho_i]$. Secondly, we see that $B_i(Q) \neq \emptyset$ for all $Q \in [\theta_i, \overline{x}_i + \rho_i]$, so on this interval the function b_i is actually well-defined. Finally, we see that $\min \tilde{R}_i(Q - \eta_i) > \eta_i$ for $\eta_i < b_i(Q)$ and $\max \tilde{R}_i(Q - \eta_i) < \eta_i$ for $\eta_i > b_i(Q)$. Graphically, this last observation means that $b_i(Q)$ corresponds to a point where $\tilde{R}_i(Q - \cdot)$ intersects with the 45°-line "from above".

Let $\theta = \max_i \theta_i$ and $\delta = \min_i [\overline{x}_i + \rho_i]$. It is clear that,

$$z(Q) = \sum_i b_i(Q),$$

is a well-defined and continuous function precisely on the interval $[\theta, \delta]$ and that $z(\theta) \geq \theta$ and $z(\delta) \leq \delta$. We have suppressed t from the previous exposition. When t is included, all of the conclusions still hold of course only now we must write $z(Q,t) = \sum_i b_i(Q,t)$ and this will be well-defined for all $Q \in [\theta(t), \delta(t)]$, where both $\theta(t)$ and $\delta(t)$ are increasing in t (that these are increasing in t follow directly from the definition of these together with the definition of a positive shock). We may without loss of generality assume that the increase in t takes place in just a single coordinate and, abusing notation slightly, we then have $t \in T = [a, b] \subseteq \mathbb{R}$ (if several of t's coordinates are increased, simply repeat the argument for each coordinate and use that in each case the aggregate will increase). It is convenient to extend $z(\cdot, t)$ such that this is defined on $[\theta(a), \delta(b)]$ for all t. We do so simply by taking $z(Q,t) = z(\theta(t), t)$ for all $\theta(a) \leq Q < \theta(t)$ and $z(Q,t) = z(\delta(t), t)$ for $I - 1 + \rho(b) \geq Q \geq \delta(t)$. Crucially, this will not introduce any new equilibrium aggregates since $z(Q,t) = z(\theta(t), t) > Q$ for all $Q < \theta(t)$, and $Q < z(Q,t) = z(\delta(t), t)$ for all $Q > \delta(t)$. We now have:

Lemma 5. *$z(Q,t)$ is increasing in t.*

Proof. Due to the way the extension of z was made above (in particular, the fact that $\theta(t)$ and $\delta(t)$ are both increasing in t), the conclusion immediately follows if we can show that each $b_i(Q,t)$ is increasing in t. $b_i(Q,t)$ corresponds to the intersection between $\tilde{R}_i(Q - \cdot, t)$ and the 45°-line where $\tilde{R}_i(Q - \eta_i)$ is strictly above (below) the 45°-line for $\eta_i < b_i(Q)$ ($\eta_i > b_i(Q)$). By assumption, t is a positive shock in the sense that the smallest and largest selections of $\tilde{R}_i(Q - \eta_i, t)$ are increasing in t (for all fixed Q and η_i). Moreover,

[32] If $\theta_i = \overline{x}_i + \rho_i$ then clearly $\rho_i < \theta_i$. In addition, $\rho_i \in \tilde{R}_i(\overline{x}_i + \rho_i - \rho_i) = \tilde{R}_i(\theta_i - \rho_i)$ hence $\theta_i, \rho_i \in B_i(\theta_i)$ contradicting that B_i is single-valued.

as shown in the proof of Theorem 1, the smallest (respectively, the largest) selection from an upper hemi-continuous correspondence with range $\mathbb{R}$ is lower semi-continuous (respectively, upper semi-continuous). In particular, the least selection is "lower semi-continuous from above" and the greatest selection is "upper semi-continuous from below". Combining we see that the correspondence $\tilde{R}_i(Q-\eta_i,t)-\{\eta_i\}$ satisfies all of the conditions of Corollary 2 in Milgrom and Roberts [27]. This allows us to conclude that $b_i(Q,t)$ is increasing in t. ■

To summarize, $Q^*(t)$ is an equilibrium aggregate given t if and only if $z(Q^*(t),t)=Q^*(t)$. In addition, we have proved that $z(Q,t)$ is continuous in Q and increasing in t. Finally, recalling the definitions of $\theta(a)$ and $\theta(b)$ from above, we have that $z:[\theta(a),\delta(b)]\times T\to\mathbb{R}$ satisfies $z(\theta(a),t)\geq\theta(a)$ and $z(\delta(b),t)\leq\delta(b)$ for all t. The conclusion of the Theorem now follows from the same argument we used at the end of the proof of the previous lemma (alternatively, it also follows from the simpler version of this result that applies to the continuous function $z(Q,t)-Q$, see *e.g.* Villas-Boas [44]).

7.3. Proof of Proposition 3

1. and *2.* were verified in the main text, so only *3.* and existence of an equilibrium remain to be addressed. To prove existence, consider the payoff function of player i after the change of coordinates $s_i\mapsto z_i=h_i(s_i)$ (for $i\in\mathcal{I}$): $\tilde{\pi}_i(z)=V_i z_i\left[R+H(\sum_{j=1}^I z_j)\right]^{-1}-\tilde{c}_i(z_i)$. Since H is convex, it is straightforward to verify that $\tilde{\pi}_i$ will be pseudo-concave in z_i under condition (21) (in particular, this condition implies that $\tilde{c}_i=c_i\circ h_i^{-1}$ will be convex). Existence of an equilibrium therefore follows from Theorem 5 (the often studied case where payoff functions are not well defined at the origin was dealt with in footnote 27). Note that a coordinate change as the one just considered does not affect any of our comparative statics results since the local solvability is ordinal (Section 4.1.2). In particular, the (uniform) local solvability will hold in this new set of coordinates if and only if it holds in the original coordinates. To prove *3.* we use Theorem 8. Note that that theorem's condition for $s_i^*(t)$ to be locally increasing in a positive shock t is

$$-[D_{s_i}\Psi_i(s_i^*,g(s^*),t)]^{-1}D_Q\Psi_i(s_i^*,g(s^*),t)\geq 0. \tag{24}$$

Since $D_{s_i}\Psi_i(s_i^*,g(s^*),t)<0$ under condition (21), (24) holds if and only if $D_Q\Psi_i(s_i^*,Q^*,t)\geq 0$ where $Q^*=g(s^*)$. For the same reason, the condition for $s_i^*(t)$ to be decreasing in t when t does not directly affect player i (the second statement of *3.*), is satisfied if and only if $D_Q\Psi_i(s_i^*,Q^*)<0$. Since $D_Q\Psi_i(s_i^*,Q^*,t)$ equals:

$$V_i\cdot\left[-\frac{h_i'(s_i^*)}{(R+Q^*)^2}+\frac{2H'\left(H^{-1}(Q^*)\right)h_i'(s_i^*)h_i(s_i^*)}{(R+Q^*)^3}-\frac{H''\left(H^{-1}(Q^*)\right)(H^{-1})'(Q^*)h_i'(s_i^*)h_i(s_i^*)}{(R+Q^*)^2}\right],$$

it is immediately seen that (24) will hold if and only if $h_i(s_i^*)\geq\eta(Q^*)$ where η is the function defined in the Theorem. By Theorem 8 the player will then increase her effort. The case where $D_Q\Psi_i(s_i^*,Q^*)<0$ and a non-affected player decreases her strategy follows by the same reasoning. ■

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