Existence, Comparative Statics, and Stability in Games with Strategic Substitutes

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First draft: August 2004. This version: December 2005 (minor corrections March 2006).

Abstract

This paper proves that pure strategy Nash equilibria (PSNE) exist in aggregative games with strategic substitutes for arbitrary strategy sets. The paper then turns to address, the structure of the equilibrium set, comparative statics, uniqueness, global stability, and existence of symmetric PSNE. The results - which to a large extend redress the literature's imbalance between games with strategic complements and substitutes - are applied to various games including multimarket oligopoly, teamwork models, and competition between teams.

Keywords: Strategic substitutes, Submodular Games, Existence of Pure Strategy Nash Equilibrium, Comparative Statics, Stability, Uniqueness, Symmetric Equilibrium, Fixed Points of Order-reversing Functions, Imperfect Competition.

JEL-classification: C72, C62, D43.

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* I am utmost thankful to Pascal Hitzler for his help during the entire lengthy process of writing this paper. The remarks and comments of Daron Acemoglu, Rabah Amir, Luis Corchón, Nikolai Kukushkin, Indra Ray, Colin Rowat, and Burkhard Schipper, have had a substaintial influence on the paper. Thanks also to seminar participants at the November 2004 UB Theory Session and at the 2005 Econometric Society World Conference. All remaining errors are my responsibility. I am particularly indebted to the late Birgit Grodal, discussions with whom were the key source of motivation for the present paper. Part of this research was carried out while the author was a post-doc enjoying the hospitality of Brown University in 2003 and 2004.

1 Introduction

A game has *strategic substitutes* if an increase in one player's strategy makes the best response of other players decrease. This outcome frequently arises under imperfect competition (strategic market games, Cournot competition, Bertrand competition, price leadership), it may reflect negative externalities at the margin (polluting firms, congestion games, the problem of the commons), it may arise if the individual agents in a team have an incentive to "free-ride" (teamwork games, public good provision), or it may result in situations with competition between teams (arms race with alliances, floor voting in political parties).[1]

Games with strategic substitutes are closely related to games with strategic complementarities (Topkis (1979), Vives (1990), Milgrom and Roberts (1990)). Both are concepts based on order theory; and a substantial advantage of both is that the strategic structure is detectable by use of the monotonicity theorem of Topkis (1978). In games with strategic complementarities, it is a direct consequence of Tarski's fixed point theorem that a pure strategy Nash equilibrium (PSNE) exists. The set of equilibria is a complete lattice (Zhou (1994)), and so is highly structured. This, under some additional assumptions on payoff functions, leads to the existence of a Pareto best and Pareto worst equilibrium (Milgrom and Roberts (1990)). Comparative statics of fixed points can be addressed with strong and easily accessible methods (Milgrom and Roberts (1990), Villas-Boas (1997)). Finally, such a question as the stability of a PSNE has found satisfactory and easily verifiable answers (Curtat (1996), Vives (2000)).

In contrast to this wealth of results, surprisingly little is known about games with strategic substitutes. In his textbook on imperfect competition, Vives summarizes the literature on existence of PSNE (without quasi-concavity) as follows: "When best replies are decreasing, existence is guaranteed for two-player games [...] but not in general. In the particular case of one-dimensional strategy sets where the best reply of a player depends only on the aggregate actions of others existence can be shown for any number of players." (Vives (2000), p.43). The existence result referred to by Vives in the case of one-dimensional strategy sets is the much celebrated result of Novshek (1985), subsequently refined by Kukushkin (1994). By the "aggregate actions of other players" is meant the linear sum of the strategies.

In a recent paper, Dubey et al. (2002) extend the Novshek-Kukushkin existence result on one-dimensional strategy sets to allow best replies to depend on a function which is not necessarily a linear sum. This extension is one example of what, following Corchón (1994) has come to be known as an *aggregative game*: An *aggregator* is an order-preserving function of all agents' strategies, and the game is *aggregative* if each player's payoff is a function of the aggregator and her own strategy.[2] All of the games mentioned in the introductory paragraph are aggregative under rather innocent assumptions (which to the best of knowledge include *all* cases studied in the applied literature).

It is the objective of this paper to redress the imbalance between games with strategic substitutes and complementarities. We shall do so by:

(i) *Proving that under a very general notion of aggregation, games with strategic substitutes possess a pure strategy Nash equilibrium regardless of the* (*dimension of*) *strategy sets*. No symmetry

[1] In section 4, examples from each of these groups are considered in detail. The terms "strategic substitutes" and "strategic complementarities" were coined by Bulow et al. (1985). Many examples of games of both types can be found in Vives (2000).

[2] Corchón (1994) states that his terminology is inspired by Dubey et al. (1980) who in a market game define the "Aggregation Axiom" to mean that the trading possibilities of an agent depends only upon the *mean* of the messages of the game's agents. For recent studies of aggregative games see Alos-Ferrer and Ania (2002), Possajennikov (2003) and Schipper (2005). Kukushkin (1994) also states that his result extends to operations which are "essentially equivalent" to addition. As will become clear, this is definitely correct if by "essentially equivalent" Kukushkin means a binary relation under which strategy sets are Abelian groups.

assumptions are imposed on agents *or* on the aggregator, so even in the one-dimensional case this considerably generalizes the above mentioned results when strategy sets are connected.[3] Moreover, we make no assumptions (except for technical ones) on the order in which joint best-replies must be decreasing, leaving us with the full flexibility of what Milgrom and Shannon (1994) call "selective ordering".

Using this existence result we are able in section 4 to claim originality when proving the existence of a PSNE in extensions of two basic models of industrial organization: The Cournot model with any number of goods and firms. And the Bertrand model with differentiated products when demand exhibit gross complementarity and/or is not symmetric.

(ii) *Characterizing the structure of the equilibrium set.* As mentioned, the set of PSNE is a lattice when there are strategic complementarities. As a consequence, players will often benefit from coordination because the extremal PSNE in the lattice - in application typically the greatest PSNE - is Pareto preferred by all agents (Milgrom and Roberts (1990)). In contrast, as we prove, the set of PSNE in a game with strategic substitutes will typically, though not always, be *completely unordered* (an antichain). From this follows that there typically is no coordination failure, so, speaking very loosely, "if someone gains, someone else stands to loose".[4] This group of results have many interesting implications, which are discussed further in the paper, and more concretely in the section with applications.

The paper then turns to:

(iii) *Comparative statics of fixed points.* If best-replies are order-preserving in an exogenous parameter, so will the extremal PSNEs be in a game with strategic complementarities (see *e.g.* Topkis (1998)). In games with strategic substitutes, we prove that a PSNE *cannot* be order-*reversing* in the exogenous parameter. Although this does provide testable implications, it is obviously not as sharp a result as the one just mentioned concerning strategic complementarities. To get sharper results more assumptions are necessarily. In symmetric equilibrium of (symmetric) games with one-dimensional strategy sets we are thus able to derive a monotonicity theorem (theorem 5) which is as sharp as any result on strategic complementarities. When the game is aggregative, Corchón (1994) in the one-dimensional case with best-replies which are single-valued contractions, has proved a number of comparative statics results all of which are as elegant in their formulation as any result on strategic complementarities. We follow up on this theme and present some more general conditions which ensure strong comparative statics conclusions.

(iv) *Uniqueness and stability.* Uniqueness and stability results for supermodular games have been derived for supermodular games by Granot and Veinott (1985) and Curtat (1996). We shall take a slightly different approach than these authors, but the results for the strategic substitutes case are as strong as any which are available on strategic complementarities.

(v) *Finally a new result providing conditions under which symmetric equilibria exist is presented.* Existence of symmetric PSNE in symmetric games with strategic substitutes is a more delicate question than in games with strategic complementarities or for that matter, in quasi-concave games (in both of these symmetric PSNE always exist in symmetric games). Nevertheless, by use of the main existence theorem in this paper described under (i) above, a very general result on existence of symmetric PSNE in symmetric games can be proved. The merit of this result is in applications where one would often like to study symmetric PSNE due to their simpler structure (an example being the

[3]The main limitation of this paper's existence result is that it does not permit discrete strategy sets as does for example Kukushkin (1994). The question about the limits of aggregation with arbitrary one-dimensional strategy sets will be persued in a separate paper. Note that all aforementioned papers on existence of PSNE implicitly or explicitly assume that aggregators are symmetric functions.

[4]Of course this "constrained optimality result", is not to be confused with a first welfare theorem. Each PSNE is Pareto optimal within the set of PSNE, *not* within the set of feasible allocations.

comparative statics result just mentioned).

A separate section (section 4) is concerned with examples and applications of our results. Among other, that section considers multimarket Cournot competition, teamwork games, and a game with competition between teams.

There are many questions which this paper can be seen as raising but not answering: Evolutionary stability (Alos-Ferrer and Ania (2002), Possajennikov (2003), Schipper (2005)), games with strategic substitutes' relationship with potential games (as raised by Dubey et al. (2002) and at least implicitly by Monderer and Shapley (1994), cf. that paper's introduction), the presence or absence of best-reply cycles (Dubey et al. (2002), Kukushkin (2004)), learning (which would be motivated by the strategic complementarities case, cf. Milgrom and Roberts (1990)), the "competitive limit" (which could be addressed in general aggregative games paralleling existing results on the Cournot model , cf. Novshek (1980)), as well as the role of mixed equilibria (which as exemplified by Acemoglu and Ozdaglar (2006) may lead to some very interesting observations).

2 General Definitions and Existence

As mentioned in the introduction, aggregative games is an active research area in its own right (see for example Alos-Ferrer and Ania (2002) and Schipper (2005)). The existence proofs in Novshek (1985), Kukushkin (1994), and Dubey et al. (2002), all fall safely within the standard notion of aggregative games. After defining games with strategic substitutes (and it is clear how this should be done), we turn in this section to define a general notion of aggregation which applies to multidimensional and asymmetric games. Since all of the mentioned studies concern the one-dimensional case, it is not equally clear how this should be done. Ultimately, the following definition is guided by mathematical necessecity in that it paves the way for the existence result of the third subsection. Having said this, the chosen formulation is actually not the most general possible but represents a suitable compromise between simplicity and generality. The more general case is discussed briefly at the end of the section. Accepting the aggregative game framework, by far the most restrictive assumption we shall make concerns the strategy sets. Essentially these must be "cubes" in $\mathbb{R}^N$, in particular they must be (compact, convex) intervals in the one-dimensional case.[5]

2.1 Games With Strategic Substitutes

Let $\mathcal{I} = \{1, \dots, I\}$ be a finite set of players. Each player has (pure) strategy set $S_i \subseteq \mathbb{R}^N$, where N is finite.[6] A typical element of S_i (a strategy) is denoted s_i. The joint strategy set is $S = \prod_{i \in \mathcal{I}} S_i$ and a joint strategy is an element $s = (s_1, \dots, s_I) \in S$. We also define $S_{-i} = \prod_{j \neq i} S_j$, with typical element s_{-i}. Each strategy set S_i comes with a partial order $\geq_i$, and the joint strategy set S is endowed with the product order so $s \geq s' \Leftrightarrow s_i \geq_i s'_i$ for all i. The orders $\geq_i$ are assumed throughout the paper to be closed, *i.e.*, each order's graph $\{(s_i, s'_i) \in S_i \times S_i : s_i \geq_i s'_i\}$ is assumed to be a closed subset of $S_i \times S_i$.[7]

[5] This is the structure of strategy sets in *smooth submodular games* which one would define analogously to Milgrom and Roberts (1990) (cf. Theorem 4) by replacing "increasing differences" with "decreasing differences". Needless to say, this class is by far the most important from an applied perspective because it is under such conditions that one can establish supermodularity and decreasing differences by looking at the second order derivatives of the payoff functions.

[6] We could have taken strategy sets to be subsets of arbitrary finite dimensional topological vector spaces without changing any of the results. In particular the assumption that the dimension of the strategy sets is the same is of no importance.

[7] All topological statements refer to the usual topology. Subsets are equipped with the induced topology and products with the product topology. That $\geq_i$ is closed is the same as saying that $s \geq_i s'$ must hold whenever $s_j \geq_i s'_j$ all j for

Denoting the payoff function of player i by $\tilde{\pi}_i : S \to \mathbb{R}$, the joint strategy $s^* \in S$ is a *pure strategy Nash equilibrium* (PSNE) if $\tilde{\pi}_i(s_i^*, s_{-i}^*) \geq \tilde{\pi}_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$ and all $i \in \mathcal{I}$. Equivalently, a PSNE is a fixed point $s^* \in R(s^*)$ of the *joint best-reply correspondence*:

$$R(\bar{s}) = arg \max_{s \in S} \sum_{i \in I} \tilde{\pi}_i(s_i, \bar{s}_{-i}) \tag{1}$$

Observe that $R = (R_i)_{i \in \mathcal{I}}$, where $R_i(\bar{s}_{-i}) = arg \max_{s_i \in S_i} \tilde{\pi}_i(s_i, \bar{s}_{-i})$ are the (individual) *best-reply correspondences.* A *selection* from R is a function $r : S \to S$, with $r(s) \in R(s)$ all $s \in S$. The selection is *order-reversing* if $s \geq_S \tilde{s} \Rightarrow r(s) \leq_S r(\tilde{s})$ for all $s, \tilde{s} \in S$. It should be clear that R has an order-reversing selection if and only if every R_i has an order-reversing selection $r_i : S_{-i} \to S_i$. A (non-cooperative pure strategy) game with strategic substitutes is defined as follows.

Definition 1 *The game $\tilde{\Gamma} = \{\mathcal{I}, (S_i, \tilde{\pi}_i)_{i \in \mathcal{I}}, \geq_S\}$ is a game with strategic substitutes if each strategy set S_i is compact, every $\tilde{\pi}_i$ is upper semi-continuous, and the joint best-reply correspondence R has an order-reversing selection.*

Any submodular and, more generally, any quasi-submodular game, certainly has best-replies which admit order-reversing selections. Indeed, in such games the greatest selection $\overline{r}(s) = \sup R(s)$, and the least selection $\underline{r}(s) = \inf R(s)$, will both exist and be order-reversing. An example of a game which has an order-reversing selection but may fail to have order-reversing extremal selections is a *weakly quasi-submodular game.*[8] In the next section we shall define (quasi-)submodular games explicitly.

Even in the one-dimensional case ($S_i \subset \mathbb{R}$) a game with strategic substitutes need not have a PSNE unless further assumptions are imposed. The following example is due to Nikolai Kukushkin.[9] Note that every selection from the each best-reply correspondence is order-reversing (so the game has strict strategic substitutes, cf. section 3).

Example 1 *Let $S_1 = S_2 = S_3 = [0, 1]$ and take:*

$$R_1(s_2, s_3) = \begin{cases} \{1\} \text{ if } s_2 < 0.5 \\ \{0,1\} \text{ if } s_2 = 0.5 \\ \{0\} \text{ if } s_2 > 0.5 \end{cases}$$

$$R_2(s_1, s_3) = \begin{cases} \{1\} \text{ if } s_3 < 0.5 \\ \{0,1\} \text{ if } s_3 = 0.5 \\ \{0\} \text{ if } s_3 > 0.5 \end{cases}$$

$$R_3(s_1, s_2) = \begin{cases} \{1\} \text{ if } s_1 < 0.5 \\ \{0,1\} \text{ if } s_1 = 0.5 \\ \{0\} \text{ if } s_1 > 0.5 \end{cases}$$

No equilibrium exists: If $s_1 = 1$ then $s_2 \leq 0.5$, so $s_2 = 0$, but then $s_3 = 1$ and so $s_1 = 0$ (likewise if $s_1 = 0$).

convergent sequences $s_j \to s$ and $s'_j \to s'$.

[8] With the obvious adaption from Shannon (1995), say that f satisfies the weak dual single-crossing in (x, t): if for all $x' \geq x$, $f(x', t) < f(x, t) \Rightarrow f(x', t') \leq f(x, t')$ for all $t' > t$. Given $s_{-i} \in S_{-i}$, the payoff function $\tilde{\pi}_i$ is *weakly quasi-supermodular* in s_i (Shannon (1995)) if for all $x, y \in S_i$: $\tilde{\pi}_i(x, s_{-i}) > \tilde{\pi}_i(x \wedge y, s_{-i}) \Rightarrow \tilde{\pi}_i(x \vee y, s_{-i}) \geq \tilde{\pi}_i(y, s_{-i})$. The game Γ is *weakly quasi-submodular* if for each $i \in \mathcal{I}$, S_i is a complete lattice, and either (1) For all $i \in \mathcal{I}$, $\tilde{\pi}_i$ is quasi-supermodular in s_i (for fixed s_{-i}), and satisfies the weak dual single crossing property in $(s_i; s_{-i})$, or (2) For all $i \in \mathcal{I}$, $\tilde{\pi}_i$ is weakly quasi-supermodular in s_i (for fixed s_{-i}), and satisfies the dual single crossing property in $(s_i; s_{-i})$.

[9] Personal communication.

2.2 Aggregative Games

To resolve the existence question pointed out at the end of the previous subsection (as well as several other questions, see the next section) we shall restrict attention to *aggregative* games. To simplify the exposition, we shall assume from now on that strategy sets are product sets, *i.e.*, $S_i = S_i^1 \times \ldots \times S_i^N \subseteq \mathbb{R}^N$, where $S_i^n \subseteq \mathbb{R}$. As a consequence we have $s_i = (s_i^1, \ldots, s_i^N)$ and each coordinate is free to vary independently of the other coordinates. Clearly this is restrictive; and at the end of this section we explain why this assumption is, in fact, substantially stronger than what is necessary (but as will become clear, relaxing it leads to some delicate topological considerations).[10]

The basic idea of aggregative games is that every player's payoff function must be a function of her own choice of strategy $s_i \in S_i$ and some *aggregate* $Q \in \mathbb{R}^M$. The aggregate depends on the joint strategy through a function $g : S \to \mathbb{R}^M$, $Q = g(s_1, \ldots, s_I)$ called the *aggregator*. Letting $X = \{g(s) : s \in S\} \subseteq \mathbb{R}^M$ (the direct image of g), we can define this formally as follows:

Definition 2 *The game $\Gamma = (\mathcal{I}, (S_i, \tilde{\pi}_i)_{i \in \mathcal{I}}, \geq_S)$ is said to admit the aggregator g, where $g : S \to X \subseteq \mathbb{R}^M$, if there exist functions $\pi_i : S_i \times X \to \mathbb{R}$, $i \in \mathcal{I}$, such that $\tilde{\pi}_i(s) = \pi_i(s_i, g(s))$, all $s \in S$, $i \in \mathcal{I}$.*

Every game admits *some* aggregator (namely the identity function $g(s) = s$). So clearly, a suitable definition of aggregative games cannot rest on this feature alone. Writing $g = (g^1, \ldots, g^M)$, the first thing we shall require is that $g^m : (S_1^j \times \ldots \times S_I^j)_{j \in \mathcal{M}(m)} \to \mathbb{R}$ where $\mathcal{M}(m) \subseteq \{1, \ldots, N\}$, $\mathcal{M}(1) \cup \ldots \cup \mathcal{M}(M) = \{1, \ldots, N\}$, and $\mathcal{M}(m_1) \cap \mathcal{M}(m_2) = \emptyset$ for all $m_1 \neq m_2$. What this says is that no two coordinate functions g^{m_1} and g^{m_2} may depend on the same coordinates from the strategy sets. This implies, in particular, that $M \leq N$ so if each $S_i \subseteq \mathbb{R}$, then $M = N = 1$, and $g : S \to \mathbb{R}$. On the other hand, M may well be strictly smaller than N (intuitively, g may "aggregate across coordinates"). The obvious example is $g(s) = \sum_i \sum_n s_i^n$, $S_i \subset \mathbb{R}^N$, where then $g : S \to \mathbb{R}_+$, so $M = 1$ and $\mathcal{M}(1) = \{1, \ldots, N\}$.

The next requirement is more complicated. When player i observes the aggregate $Q \in X$ in response to some *reference strategy* $\theta_i \in S_i$, we assume that she is able to correctly anticipate the new aggregate Q which results if she chooses a different strategy $s_i \in S_i$, while the other players' strategies $\bar{s}_{-i} \in S_{-i}$, remain fixed. Formally, this will be the case if (and only if) there exists a function $F_i : X_{-i} \times S_i \to X$, where $X_{-i} = \{g(\bar{s}_{-i}, \theta_i) : s_{-i} \in S_{-i}\} \subset X$, such that:

$$g(\bar{s}_{-i}, s_i) = F_i(g(\bar{s}_{-i}, \theta_i), s_i), \text{ for all } s_i \in S_i \text{ (and } \bar{s}_{-i} \in S_{-i} \text{ fixed)} \tag{2}$$

We may also write (2) in terms of g's coordinates:

$$g^m(\bar{s}_{-i}^{\mathcal{M}(m)}, s_i^{\mathcal{M}(m)}) = F_i^m(g^m(\bar{s}_{-i}^{\mathcal{M}(m)}, \theta_i^{\mathcal{M}(m)}), s_i^{\mathcal{M}(m)}) \ , m = 1, \ldots, M \tag{3}$$

where then $s_i^{\mathcal{M}(m)} \in S_i^{\mathcal{M}(m)}$ (and $\bar{s}_{-i}^{\mathcal{M}(m)} \in S_{-i}^{\mathcal{M}(m)}$ fixed). Note that $F_i^m : X_{-i}^m \times S_i^{\mathcal{M}(m)} \to X^m$ where $X_{-i}^m = \{g^m(\bar{s}_{-i}^{\mathcal{M}(m)}, \theta_i^{\mathcal{M}(m)}) : s_{-i}^{\mathcal{M}(m)} \in S_{-i}^{\mathcal{M}(m)}\} \subset X^m = \{g^m(s^{\mathcal{M}(m)}) : s^{\mathcal{M}(m)} \in S^{\mathcal{M}(m)}\} \subseteq \mathbb{R}$.

In the next statement and throughout, $X \subseteq \mathbb{R}^M$ is equipped with the usual order $\geq$ (so the order on X does not necessarily stand in any specific relationship with $\geq_S$, see example 4 below). When two elements are comparable $x \geq y$ but not identical, $x \neq y$, we write $x > y$. The aggregator g is *strictly order-preserving* if $s >_S s' \Rightarrow g(s) > g(s')$.

Definition 3 *The game $\Gamma = (\mathcal{I}, (S_i, \tilde{\pi}_i)_{i \in \mathcal{I}}, \geq_S)$ is aggregative if it admits and aggregator $g : S \to X$ which is continuous, strictly order-preserving, and satisfies the two previous requirements (the coordinate-condition immeditately after definition 3, and condition (2) for all $i \in \mathcal{I}$).*

[10] We could have taken S_i to be a finite product of chains without changing anything.

By (2), $\tilde{\pi}_i(s_i, \bar{s}_{-i}) = \pi_i(s_i, g(\bar{s}_{-i}, s_i)) = \pi_i(s_i, F_i(g(\bar{s}_{-i}, \theta_i), s_i))$ in an aggregative game. It follows that each player's best replies will be a function of $g(\bar{s}_{-i}, \theta_i) \in X_{-i}$. Intuitively $g(s_{-i}, \theta_i)$ is the "market signal" the agent observes upon sending her reference strategy θ_i to the market (and it is clear that she needs no more information to make a decision because by (2) she can compute the aggregate $g(s)$ for any other strategy $s_i \in S_i$). We shall often write $R_i(g(\bar{s}_{-i}, \theta_i))$ for the individual best-replies of player $i \in \mathcal{I}$ to stress this.

Example 2 (Separable functions are aggregators) *If $G(g(s)) = \sum_i G_i(s_i)$, where $G_i = (G_i^1, \ldots, G_i^M)$, $i \in \mathcal{I}$, $G_i^m : S_i^m \to \mathbb{R}$, are order-preserving functions, and G is continuous with a continuous and order-preserving inverse, then g and $F_i(x, s_i) = G^{-1}(x + G_i(s_i))$ satisfy* (2) *for all i and $h_i(Q, s_i) = G(Q) - G_i(s_i)$ which clearly is order-preserving in Q and order-reversing in s_i. Separable aggregators are important of the simple reason that these are the ones most often encountered in applications (see section 4). Not every aggregator is separable, however, as the next example shows.*

Example 3 (Building Aggregators Recursively) *Let $F_i : X_{-i} \times S_i \to X$, $i = 1, \ldots, I$ be a sequence of functions which are (i) continuous and strictly order-preserving, (ii) continuous, (iii) admit $\theta_i \in S_i$ such that $F_i(x, \theta_i) = x$ for all $x \in X_{-i}$. Then $g(s_1, \ldots, s_I) = F_1(F_2((\cdots F_I(g(\theta_1, \ldots, \theta_I), s_I), \cdots), s_2), s_1)$ is a regular aggregator. Conversely, if g is a regular aggregator each F_i must satisfy (i)-(iv) (see lemma* **??** *and surrounding discussion in the proof of the main theorem). As may be checked, the ordering in this recursive construction does not matter, so we could instead have taken, say, $F_I(F_{I-1}((\cdots F_1(g(\theta_1, \ldots, \theta_I), s_1), \cdots), s_{I-1}), s_I)$ and this would produce the same aggregator g. As a specific example which is not separable, let $F_i = F$ and $S_i = S$ a compact subset of $\mathbb{R}_+$ for all i and $X \subseteq \mathbb{R}$. Take $F(s_1, s_2) = s_1 + s_2 + s_1 s_2$ which for $I = 3$ yields $g(s_1, s_2, s_3) = s_1 + s_2 + s_3 + s_1 s_2 + s_1 s_3 + s_2 s_3 + s_1 s_2 s_3$ (Dubey et al. (2002)). Here $\theta_i = 0$ and $h = h_i$ (the inverse) is given by $h(Q, s_i) = \frac{Q - s_i}{1 + s_i}$. We shall apply this aggregator, and its multidimensional extension in the teamwork game of section 4.*

The next example shows that in applications one enjoys the full flexibility of what Milgrom and Shannon (1994) call *selective ordering*. As will be explored in details in the context of games with competition between teams in section 4, strategic substitutes may - as may strategic complementarities - be "hidden" in the sense that it applies only if one, or both, of the orders $\geq_S$ and $\geq$ are not chosen as the usual ones.[11]

Example 4 *Let $I = 3$, $S_i \subseteq X \subseteq \mathbb{R}^N$ and take $g(s) = \alpha_1 s_1 - \alpha_2 s_2 + \alpha_3 s_3$, where the α's are positive constants. Here $F_1(x, s_1) = x + \alpha_1 s_1$, $F_2(x, s_2) = x - \alpha_2 s_2$, and $F_3(x, s_3) = x + \alpha_3 s_3$ clearly satisfy* (2)*. However, g is not order-preserving if $\geq_S$ is the usual order. But, defining $\geq_S$ by virtue of "$s \geq_S \tilde{s} \Leftrightarrow s_1 \geq \tilde{s}_1$, $s_2 \leq \tilde{s}_2$, $s_3 \geq \tilde{s}_3$", where $\geq$ is the usual order on $\mathbb{R}^N$, g will be order-preserving, and since it is separable it is therefore an aggregator.*

2.3 Existence of PSNE

Armed with a proper understanding of aggregative games, we are now ready to state the main result of this paper. Just as Novshek (1985) and Kukushkin (1994), the proof focuses on the backward reply correspondence. In the present set-up this is defined as $B = (B_i)_{i \in \mathcal{I}}$ where $B_i : X \to 2^{S_i} \cup \emptyset$, $i \in \mathcal{I}$, are the individual backward reply correspondences:

$$B_i(Q) = \{s_i \in S_i : s_i \in R_i(h_i(Q, s_i))\} \tag{4}$$

[11] For a number of results and examples on the use of non-Euclidean orders see Jensen (2004).

We have $B : X \to 2^S \cup \emptyset$ under the convention that $B(Q) = \emptyset$ whenever $B_i(Q) = \emptyset$ for *some* $i \in \mathcal{I}$. Except for the added generality in terms of strategy sets and aggregators, (4) is the direct parallel to the backward reply correspondence first explored by Selten (1970). In particular, $s^* \in S$ will be a PSNE if and only if there exists $Q \in X$ such that $g(s^*) = Q$ and $s^* \in B(Q)$.

Theorem 1 (Existence of PSNE) *Let Γ be an aggregative game with strategic substitutes. Then there exists a pure strategy Nash equilibrium under the following conditions:*
(i) Each strategy set is a product set $S_i = \prod_{n=1}^N S_i^n \subseteq \mathbb{R}^N$, where S_i^n is a connected subset of $\mathbb{R}$.
(ii) There exists $Q^* \in X$, $B(Q^*) \neq \emptyset$, such that $g(s) \leq Q^*$ for all $s \in S$.

Remark 2.1 *If $\geq_S$ is the usual order so that S is a complete lattice with greatest element $\top_S$; then (ii) is satisfied provided that $B(g(\top_S)) \neq \emptyset$.*

Condition *(i)* is clearly restrictive, effectively ruling out discrete strategy sets. If $N = 1$ (so strategy sets are subsets of the reals), it implies that S_i is a closed interval. *(ii)* places an upper bound on the realizations of the aggregator: If s is a feasible joint strategy, its realization under the aggregator must be below Q^*. For example, imagine that $g(s) = \sum_i s_i$ and that every S_i has a greatest element $\top_i$. Taking $Q^* = \sum_i \top_i$ and assuming that $\{0\} = R_i(\sum_{j \neq i} \top_j)$ for all i, *(iii)* will be satisfied because $0 \in B(Q^*)$ while clearly $g(s) \leq Q^*$ for all $s \in S$.[12]

The fact that theorem 1 is the first to answer such a question as when a pure strategy equilibrium exists in multimarket oligopolistic models without quasi-concavity, is perhaps indicative of its proof's complexity. Because of this, an attempt will now be made at explaining it in some detail.

The proof begins by an application of Zorn's lemma in order to find a selection f from B defined upon a maximal subset of X, call it P^*, which is an up-set with greatest element Q^* of (ii) above.[13] Being an upset means that if $x \in P^*$ and $y \geq x$, $y \in X$, then $y \in P^*$. As it turns out, P^* becomes topologically very well-behaved (an absolute retract, in fact). In particular, it is contractible and contains its "lower boundary" ∂P^* defined as the set of minimal elements in P^*. ∂P^* will always be a set of dimension less than or equal to $M - 1$, M being the dimension of $g(S)$ (the direct image of g). When $g(S)$ is one-dimensional as is the case in Novshek and Kukushkin's proofs; ∂P^* consists of a single point (the infimum of P^*).[14] This point corresponds exactly to Novshek's termination point, a point which Kukushkin denotes by t^0 (Kukushkin (1994), p.24). Unfortunately, ∂P^* will generally not have an infimum when the dimension is greater than one, and more importantly, it is not so that every element in ∂P^* is a fixed point of the backward reply correspondence. Thus from this point and on, there ceases to be any connection with the proofs of Novshek and Kukushkin.

Any $Q \in P^*$ determines a joint backward reply $f(Q) \in B(Q)$ by the selection f mentioned above. Defining $F(Q) = Q - g(f(Q))$, a PSNE consequently exists provided that $F(Q^*) = 0$ for some $Q^* \in P^*$ (in the following discussion we are taking $X \subseteq \mathbb{R}^M$ with the usual order). By the way the set P^* is defined, it will always hold that $F(Q) \geq 0$. Our first important observation concerning F is that it can *never* hold that $F(Q) \gg 0$ (lemma 11) when $Q \in \partial P^*$. Thus $F : \partial P^* \to \partial \mathbb{R}^M_+$. The

[12] One might conjecture that it would always be possible to suitably "extend" strategy sets such that the previous description applies. In most applications this is probably so, but it is *not* true as a general statement as returned to below.

[13] This step is similar to the argument used by Smithson (1971) to prove the existence of fixed points of order-preserving multifunctions. Lemma 6 can be seen as the verification of our equivalent to Smithson's "Condition III". By a similar argument one can show that Smithson's Condition III will be satisfied provided that the correspondence is upper hemi-continuous (while both conditions I and II are in fact satisfied if the correspondence is increasing in the strong set order).

[14] If payoff functions are additively separable, this will be the case more generally and then the proof is essentially over. In general, P^* does not have an infimum unless X is one-dimensional or payoffs are additive.

second important observation is that in ∂P^* are points which map to every axis of $\mathbb{R}^M_+$ (so there is one point with $F^1(Q) = 0$, one with $F^2(Q) = 0$, and so forth). If for example $M = 3$, there are three such points (one for each axis). There exists a continuous map $t : \mathbb{S}^{M-2} \to \partial P^*$ which crosses each of these points once. So if $M = 3$, $\mathbb{S}^{M-2} = \mathbb{S}^1$ (the unit circle), so t is a loop in ∂P^*.[15] Because all homotopy groups of ∂P^* are trivial (it is contractible), the map t will be null-homotopic, *i.e.*, the map t will be homotopic with a constant map. In the three-dimensional loop-case mentioned, this intuitively means that the loop can be continuously deformed to a point.

The final piece of information we need about F is that this will be continuous in a certain topology, which is essentially the so-called Scott-topology, cf. Gierz et al. (2003), chapter 2 (the qualifier is because we need to be carefull in its definition at the boundary ∂P^*). This topology is not a Hausdorff topology (so a convergent sequence may have multiple limits); and it certainly does not rule out the possibility of "jumps". It does, however, rule out a certain kind of jumps; and precisely those which would be necessary for the composition $F \circ t : \mathbb{S}^{M-2} \to \mathbb{R}^N_+$ to avoid passing 0 when it is deformed to a point. Leaving the technicalities, note that it is easy to see that a loop in $\mathbb{R}^3_+$ with the properties described above could never be continuously deformed to a point without passing the origin if the topology is the usual one. This basic intuition carries over to the non-Hausdorff framework; and so we end up concluding that there must exist $Q \in X$ with $F(Q) = 0$ which is to say that there exists a pure strategy Nash equilibrium.

We end this section we an example which shows that the part of assumption *(ii)* which says that $B(Q^*) \neq \emptyset$ is critical. Indeed, if one does not make sure that the backward reply correspondence is well-defined at least at one point $Q \in X$; non-existence of a PSNE is inevitable. This example is motivated by discussions with Nikolai Kukushkin.[16]

Example 5 *Let* $S_1 = [0,1] \times \{0\} \times \{0\}$, $S_2 = \{0\} \times [0,1] \times \{0\}$, *and* $S_3 = \{0\} \times \{0\} \times [0,1]$ *and* $g(s) = s_1 + s_2 + s_3$ *(which is a regular aggregator). Letting* $s_1 = (t_1, 0, 0) \in T_1$, $s_2 = (0, t_2, 0)$, $s_3 = (0, 0, t_3)$, *note that* $g(s) = (t_1, t_2, t_3)$. *The game is aggregative provided that:* $\tilde{\pi}_i(s_i, s_{-i}) = \pi_i(s_i, g(s)) = \pi_i((t_1, 0, 0), (t_1, 0, 0), (0, t_2, 0), (0, 0, t_3)) = f_i(t_1, t_2, t_3)$. *But then the game is automatically aggregative and, furthermore,* (s_1^*, s_2^*, s_3^*) *is a PSNE if and only if* (t_1^*, t_2^*, t_3^*) *is a PSNE in the one-dimensional game with payoffs* f_i *and strategies* $t_i \in T_i = [0,1]$. *Clearly, a PSNE does not necessarily exist because the "one-dimensional reduction" is not required to be aggregative (see example* 1*). What happens here is that (iii) is being violated because* $B(Q) = \emptyset$ *for all* $Q \in g(S) = [0,1]$. *One can check this point-by-point in any concrete case; but it is seen generally by the following: For* $s \in B(Q)$, $s_i \in R_i(Q - s_i)$ *for all* i *where* $Q - s_i \in X_{-i}$ *(here* $X_{-1} = [(0,0,0),(0,1,1)]$, $X_{-2} = [(0,0,0),(1,0,1)]$, *and* $X_{-3} = [(0,0,0),(1,1,0)]$*). But* $Q - s_i \in X_{-i}$ *for* $i = 1,2,3$ *if and only if* $Q = (s_1^1, s_2^2, s_3^3)$. *Thus* $s \in B(Q)$ *if and only if (i)* $Q = (s_1^1, s_2^2, s_3^3)$ *and (ii)* $Q^i \in R_i^i((0^i, Q^{-i}))$. *But this exactly brings us to the "one-dimensional reduction"; and so there exists a joint best-reply if and only if the reduction has a PSNE.*

[15] The case $M = 2$ is a little special. Here we have $t : [0,1] \to \partial P^*$ with the end-points corresponding to the two points just mentioned. Of course this is the same as saying that those points can be connected with a continuous path.

[16] Novshek (1985) solves the "boundary problem" in the spirit of general equilibrium theory by assuming defining best-replies on the whole of $\mathbb{R}_+$ and imposing an upper bound such that all firms will choose zero output if the sum of the other players' outputs is at or above this bound. Kukushkin's extension argument at the beginning of the proof of Proposition 1 (Kukushkin (1994), p.24) makes it possible to dispense with any such uniform upper bound if in stead strategy sets are assumed to be compact. The relationship with the multidimensional case is subtle: If one is able to extend strategy sets and best-replies in the manner described after theorem 1, *(ii)* is indeed implied. Here one uses the fact that a correspondence $R : S \to 2^S$ has a fixed point if and only if any correspondence $\tilde{R} : T \to 2^S$, $S \subseteq T$, $\tilde{R}_{|S} = R$, has a fixed point. But one may not "extend" so as to redefine the best-reply map (specifically, in the example, one would have to allow best-replies to be a funcion of the firms' *own* strategies in order to make the backward reply correspondence non-empty at $Q^* = (1,1,1)$).

3 Properties of Equilibria

Having at this point established that if a game with strategic substitutes is aggregative it has a pure strategy Nash equilibrium (PSNE), we are ready to explore the resulting family of games' properties. With a keen eye on applications, this section characterizes the set of PSNE from various perspectives, including the structure of the equilibrium set, comparative statics, uniqueness, and stability. In addition, several results concerning symmetric games will be presented. As usual a game is symmetric if players share the same strategy set and payoff function (the latter being up to a monotone transformation). Additionally, as in Milgrom and Roberts (1990), it is required that the order $\geq_S$ has the form $\geq_{S_i} \times \ldots \times \geq_{S_i}$ where $\geq_{S_i}$ is an order on the strategy set. In an aggregative game, symmetry of payoff functions obviously implies that the aggregator g is a symmetric function.

While some of the following results apply to general games with strategic substitutes, some of them hold only if the game is aggregative, and in some cases submodular.[17] In submodular games, the joint strategy set S is assumed to be a lattice, *i.e.*, $s^1 \wedge s^2, s^1 \vee s^2 \in S$, for all $s^1, s^2 \in S$. Here $s^1 \wedge s^2$ is the infimum and $s^2 \vee s^2$ the supremum of s^1 and s^2. Moreover, the joint objective function $\tilde{\pi}(s, \bar{s}) = \sum_{i \in I} \pi_i(s_i, g(\bar{s}_{-i}, s_i))$ is assumed to be supermodular in $s \in S$ and have decreasing differences in $(s, \bar{s}) \in S \times S$. Under the additional requirements that S is compact and $\tilde{\pi}$ is upper semi-continuous (both of which feature as part of our general definition of a game with strategic substitutes, definition 1), it follows that the joint best-reply map R will be non-empty lattice valued, have a closed graph, and be *descending*, *i.e.*, $s^1 \geq s^2$, $\bar{s}^1 \in R(s^1)$, $\bar{s}^2 \in R(s^2)$ implies $\bar{s}^1 \wedge \bar{s}^2 \in R(s^1)$ and $\bar{s}^1 \vee \bar{s}^2 \in R(s^2)$. This in turn implies that the greatest and least selections are order-reversing. We remark that this also holds if the game is quasi-submodular, but to keep things simple attention will be restricted mainly to submodular games in what follows.

3.1 The structure of the equilibrium set

In a game with strategic complementarities the set of PSNE (the equilibrium set) will be a complete lattice (Zhou (1994)). From this follows readily that if payoffs satisfy a certain, usually very reasonable, monotonicity requirement then the greatest fixed point *Pareto dominates* any other fixed point, while the least fixed is *Pareto dominated* by any other fixed point (Milgrom and Roberts (1990)). As we now argue, such results generally do not hold in games with strategic substitutes for here the tendency is rather for the set of equilibria to be completely unordered (an antichain).

Consider first the special case where R is *strongly order-reversing*, *i.e.*, where $s^1 \leq_S s^1$ implies that $\bar{s}^1 \geq_S \bar{s}^2$ for all $\bar{s}^1 \in R(s^1)$ and $\bar{s}^2 \in R(s^2)$. When payoffs are strictly quasi-concave in own strategies, and more generally whenever best-replies are singletons, the singleton joint best-reply correspondence is of course strongly order-reversing. This includes the model studied by Corchón (1994). If one assumes that every player acts according to some "selection criteria" and that the persued selection is order-reversing (*e.g.*, he may always choose the greatest best-reply in a submodular game); it is clear that the following result also applies.[18]

Theorem 2 *Let $R : S \to 2^S$ be a strongly order-reversing correspondence. Then there cannot exist two (different) fixed points which are ordered by $\geq_S$.*

Proof: *Pick $s^1 <_S s^2$ such that $s^1 \in R(s^1)$, and $s^2 \in R(s^2)$. But by the hypothesis then, $s^2 \leq_S s^1$, which is a contradiction.* □

[17] Note that in this paper a game has strategic substitutes if its joint best-reply correspondence has an order-reversing selection. This is a strictly weaker requirement than that the game be (quasi)-submodular.

[18] An example of such a "selection criteria" is Karp et al. (2003) who when discussing pure strategy Nash equilibria in a game where agents face a choice between going to a bar or staying home: "assume that an individual who is indifferent between the two actions stays home" (p.3).

Theorem 2 says that if the joint best-reply correspondence is strongly order-reversing, the set of Nash equilibria is an *antichain*. Yet another way of stating this is in the language of general equilibrium theory: Say that a PSNE, s^1, is *constrained efficient* if there does not exist another PSNE, s^2, such that $s^2 >_S s^1$. Theorem 2 then says that any Nash equilibrium is constrained efficient. Obviously, one cannot possibly hope to find reasonable conditions under which equilibria are Pareto ranked in such a situation. Unfortunately, one should be carefull when appealing to strong order-reversion of R (the joint best-reply correspondence). If R is strongly order-reversing so is every R_i; but the converse of this statement is false. It follows that a game with strict strategic substitutes (every R_i is strongly order-reversing) is not covered by theorem 2.

Example 6 *Let there be two players, the first having a singleton strategy set $S_1 = \{s_1\}$ and the second a strategy set consisting of an ordered pair of elements, $S_2 = \{s_2^1, s_2^2\}$. Assume that agent 2's payoff function is constant so that $R_2(s_1) = \{s_2^1, s_2^2\}$. Of course the first player has no actual choice, $R_1(s_2^1) = R_1(s_2^2) = \{s_1\}$, but the first player's choice may still affect the payoff, say, $\tilde{\pi}_2(s_1, s_2^1) < \tilde{\pi}_2(s_1, s_2^2)$. Clearly both best-reply correspondences are strongly order-reversing. There are two fixed points, $(s_1, s_2^1) <_S (s_1, s_2^2)$. Finally, there is a coordination problem in the sense that (s_1, s_2^2) Pareto dominates (s_1, s_2^1). R has a strictly increasing selection, $R(s_1, s_2^1) = (s_2^1, s_1)$ going to $R(s_1, s_2^2) = (s_2^2, s_1)$; so R is not strongly order-reversing.*

As mentioned above, one may go on to assume that every agent's choice is guided by some selection rule, *i.e.*, that every agent is described by an order-reversing best-reply selection r_i from her best-reply correspondence R_i. In this case, the set of equilibria will be an antichain (and non-empty if the selection is continuous from below, as is the greatest selection of a submodular game). Still, this is not always a satisfactory modelling device, and we shall therefore persue the question further. Restricting attention to submodular games we are able to see clearly how any ordered fixed points may potentially arise. Of course we are also able to use the following result to rule out such ordered equilibria.

Theorem 3 *Let Γ be a submodular game and let $s^2 >_S s^1$ be two ordered PSNE. Then for every agent i, the payoff function must be additive on $[s^1, s^2]$, i.e., $\tilde{\pi}_i(s_i; s_{-i}) = u_i(s_i) + v_i(s_{-i})$ for $s \in [s^1, s^2]$ where u_i and v_i are real-valued functions.*

Proof: *Take $s^1 \in R(s^1)$ and $s^2 \in R(s^2)$ with $s^2 >_S s^1$. It must necessarily be the case that $\{s^1, s^2\} \subseteq R(s^1)$ and $\{s^1, s^2\} \subseteq R(s^2)$. In terms of the payoffs, for every i: $\tilde{\pi}_i(s_i^2; s_{-i}^1) = \tilde{\pi}_i(s_i^1; s_{-i}^1) \geq \tilde{\pi}_i(t_i; s_{-i}^1)$, $t_i \in S_i$. Likewise, $\tilde{\pi}_i(s_i^2; s_{-i}^2) = \tilde{\pi}_i(s_i^1; s_{-i}^2) \geq \tilde{\pi}_i(t_i; s_{-i}^2)$, all $t_i \in S_i$. Since every $\tilde{\pi}_i$ has decreasing differences in (s_i, s_{-i}), $\tilde{\pi}_i(s_i^2, s_{-i}) - \tilde{\pi}_i(s_i^1, s_{-i})$ is non-increasing in s_{-i} and so $\tilde{\pi}_i(s_i^2, t) = \tilde{\pi}_i(s_i^1, t) \equiv v_i(t)$ for all $t \in S_{-i}$ with $s_{-i}^1 \leq_S t \leq_S s_{-i}^2$. Next note that $\tilde{\pi}_i(s_i^2; s_{-i}^2) - \tilde{\pi}_i(t_i; s_{-i}^2) = \tilde{\pi}_i(s_i^1; s_{-i}^2) - \tilde{\pi}_i(t_i; s_{-i}^2)$ hence by decreasing differences, this difference is constant in s_{-i}^2 whenever $s_i^1 \leq t_i \leq s_i^2$, i.e, $\tilde{\pi}_i(s_i^2; t_{-i}) - \tilde{\pi}_i(t_i; t_{-i}) = \tilde{\pi}_i(s_i^1; t_{-i}) - \tilde{\pi}_i(t_i; t_{-i}) \equiv -u_i(t_i)$ for all $t_{-i} \in S_{-i}$. It now follows that: $\tilde{\pi}_i(t_i; t_{-i}) = \tilde{\pi}_i(s_i^1; t_{-i}) + u_i(t_i) = v_i(t_{-i}) + u_i(t_i)$.* □

Recall that $\tilde{\pi}_i(s_i, s_{-i})$ has strictly decreasing differences in (s_i, s_{-i}) if $\tilde{\pi}_i(s_i', s_{-i}) - \tilde{\pi}_i(s_i, s_{-i})$ is strictly decreasing in s_{-i} whenever $s_i' > s_i$. On the other hand, $\tilde{\pi}_i(s_i', s_{-i}) - \tilde{\pi}_i(s_i, s_{-i}) = u_i(s_i') - u_i(s_i)$ with the additive form of theorem 3 and so it follows:

Corollary: *Assume that Γ is a strictly submodular game (every payoff function has strictly decreasing differences in (s_i, s_{-i})). Then there cannot be two equilibria $s^{*,1}$ and $s^{*,2}$ with $s_i^{*,2} > s_i^{*,1}$ and $s_j^{*,2} > s_j^{*,1}$ for $j \neq i$, i.e., the equilibria must be identical except for the entries of at most one* *player.*

It should be mentioned that the previous results do not require the game to be aggregative. Economically, theorem 2 expresses that *coordination failures are absent*: Even if the players were allowed to meet and choose a PSNE among the entire set of PSNE, they would generally not be able to agree unambiguously on a preferred equilibrium. This sharply contrasts with supermodular games where, under the monotonicity conditions mentioned above, the coordination failure is severe in the sense that whenever there are multiple equilibria and agents' payoff functions are able to rank ordered strategies, the greatest PSNE will be preferred by *every* agent. In the case of theorem 3, the statement is weaker. Here the player, say the i'th, whose entries differ in any ordered pair of equilibria will necessarily be indifferent between the two, so no strict Pareto improvement is ever possible. However, if all payoff functions are decreasing (or increasing) in the other player's strategies; the resulting equilibria will be Pareto ranked and in this case there is definitely a failure of coordination. Turning this around, if some players' payoff functions are increasing and others' decreasing in player i's strategy; once again ordered fixed points cannot be Pareto ranked.

3.2 Comparative Statics

In models with strategic substitutes, comparative statics of fixed points is much more challenging mathematically than in games with strategic complementarities (for the latter see *e.g.* Vives (2000)). The situation we shall investigate here is when an exogenous parameter $t \in T$, where $(T, \geq_T)$ be a poset (the parameter space), affects the joint best-reply map so that we have $R : S \times T \to 2^S$. Letting $\text{Fix}(t) = \{s \in S : s \in R(s,t)\}$, the question is whether the correspondence $\text{Fix} : T \to 2^S$ has "nice" monotonicity properties (*e.g.*, all selections could be order-preserving). The first result, which is very general, is related to theorem 3 in Villas-Boas (1997).[19] If $t^2 >_T t^1$ implies that $\bar{s}^2 \geq_S \bar{s}^1$ for all $\bar{s}^1 \in R(s,t^1)$ and $\bar{s}^2 \in R(s,t^2)$ (for every fixed $s \in S$), we say that R is strongly order-preserving in t. In a parameterized family of submodular games, this outcome is ensured if each of the payoff functions, $\tilde{\pi}_i(s_i, s_{-i}, t)$, $t \in T$, has strictly increasing differences in $(s_i, t) \in S_i \times T$.

Theorem 4 (Comparative statics is not counterintuitive) *Assume that R is strongly order-preserving in t and that for all $s \in R(s^1,t)$ and any $s^2 \geq_S s^1$ there exists $y \in R(s^2,t)$ with $y \leq_S s$. Pick $t^1, t^2 \in T$ with $t^2 >_T t^1$ and $s^1 \in \text{Fix}(t^1)$, $s^2 \in \text{Fix}(t^2)$. Then it cannot happen that $s^1 >_S s^2$.*

Proof: *Assume to the contrary, that is pick $s^1 \in R(s^1,t^1)$, $s^2 \in R(s^2,t^2)$, $s^1 >_S s^2$. Then $z \geq_S s^1 >_S s^2 \geq_S y$ for all $z \in R(s^1,t^2)$ and some $y \in R(s^1,t^2)$, which is a contradiction.* □

Remark *As is seen from the proof, the assumptions on R may be replaced by: R is strongly order-reversing in s; and for all $s \in R(s^1,t^1)$ and any $t^2 \geq_T t^1$ there exists $y \in R(s^1,t^2)$ with $y \geq_S s$.*

Theorem 4 transfers to submodular games as follows:[20]

Corollary (Comparative Statics in Submodular Games) *Let Γ be a submodular game with parameterized payoff functions $\tilde{\pi}_i(s_i, s_{-i}, t)$, $t \in T$, which have strictly increasing differences in $(s_i,t) \in S_i \times T$. Then the conclusion of theorem 4 is valid, i.e., when $t^2 \geq_T t^1$ it cannot happen that $s_1 >_S s_2$ for $s^1 \in \text{Fix}(t^1)$, $s^2 \in \text{Fix}(t^2)$.*

[19]Villas-Boas' result concerns the case where best-replies are single-valued.

[20]These results, of course, remain valid when substituting quasi-submodular for submodular and dual single crossing property for decreasing differences.

Proof: *By submodularity, there exists an order-reversing least section,* $\underline{r} : S \to S$, $\underline{r}(s,t) \in R(s,t)$ *for any fixed* $t \in T$. *Taking* $s^1 \leq_S s^2$ *and* $s \in R(s^1, t)$, *then* $y := \underline{r}(s^2) \leq_S \underline{r}(s^1) \leq_S s$, *which shows that the second hypothesis of the theorem is satisfied. The first (R strongly order-preserving in t) follows from the discussion prior to the theorem.* □

The corollary's conclusion may be compared to the situation in supermodular games.[21] With ascending best-replies, the fixed point correspondence will have a greatest and least selection and both of these will unambiguously be order-preserving functions of $t \in T$ provided that payoffs have increasing differences in (s_i, t) (see *e.g.* Vives (2000)). The result above says only that $t^2 >_T t^1$ will imply $s^2 \not<_S s^1$ so if, for example, there is a positive shock to demand in the Cournot model, this *cannot* lead to a new PSNE where all firms' output have fallen. On the positive side, the present result is valid for *any* selection from the fixed point correspondence $\mathrm{Fix} : T \to 2^S$, while in the supermodular case there may be selections which are order-reversing over some interval.

Sometimes, one is able to use the previous result to derive very sharp comparative statics conclusions. In symmetric equilibrium (to be examined further in the next subsection), this is the case if strategy sets are totally ordered.

Theorem 5 (Comparative Statics in Symmetric Games with Totally Ordered Strategy Spaces) *Let* Γ *be a symmetric submodular game with totally ordered strategy spaces which is parameterized by* $t \in T$ *so that payoff functions have strictly increasing differences in* $(s_i, t) \in S_i \times T$. *Let* $\mathrm{Fix}^{sym}(t)$ *denote the set of symmetric PSNE given* $t \in T$. *Then* $t^1 \leq_T t^2$ *implies that* $s^1 \leq_S s^2$ *for all* $s^1 \in \mathrm{Fix}^{sym}(t^1)$ *and* $s^2 \in \mathrm{Fix}^{sym}(t^2)$, *i.e., the correspondence* Fix^{sym} *is strongly order-preserving.*

Proof: *By theorem 4* $s^1 \not>_S s^2$, *hence by the symmetry assumption* $s^1 \leq_S s^2$. □

In theorem 5 the exogenous shock is implicitly assumed to occur symmetrically across the agents. In the language of Corchón (1994), this is a *generalized shock*. One application is that of Vives (2000), section 4.3.1., where t is an excise tax on costs $C = C(s_i, t)$, in the Cournot model. Theorem 5 shows that in order to reach strong comparative statics conclusions it is unnecessary to assume that best-replies are single-valued contractions as done by Vives (2000).

If a shock is *idiosyncratic*, *i.e.*, if it affects only one of the agent's payoff function, intuition suggests that a positive shock should make the affected agent's strategy increase and the others' decrease. In a two-player submodular game with the product order, this is indeed the case.[22]

In the case of an idiosyncratic shock, we have $R : S \times T_i \to 2^S$ where for some agent i, $R_i = R_i(s_{-i}, t_i)$ and for all other agents $j \neq i$, $R_j = R_j(s_{-j})$. Assume to simplify notation, that the idiosyncratic shock affects agent $i = 1$. Consider an order-reversing selection $r : S \times T_1 \to S$ from R (which is order-preserving in $t_1 \in T_1$). Pick s^0 such that $r(s^0, t_1) = s^0$ (a fixed point). Then increase t_1 to t_2, and let $s^1 = r(s^0, t_2)$. We note that $s_1^1 \geq s_1^0$ while $s_j^1 = s_j^0$ for $j \neq 1$. This implies that in the next step of the iteration, *i.e.* for $s^2 = r(s^1, t_2)$, $s_1^2 = s_1^1$ and $s_j^2 \leq s_j^1$ for $j \neq 1$. So in the next step, $s^3 = r(s^2, t_2)$, $s_j^3 \geq s_j^2$ for all j, and so $s_j^4 \leq s_j^3$, for all j, etc. From $\tau = 0$ to $\tau = 2$, s_1^τ increases, and from $\tau = 0$ to $\tau = 2$, s_{-1}^τ decreases. Most likely, this is what our intuition is able to grasp. But thereafter, *i.e*, for $\tau > 2$, s^τ will *oscillate*.[23] Though both the first- and the second-order effects have

[21] Again, quasi-supermodularity in the sense of Milgrom and Shannon (1994) is of course equally good.

[22] As is well known, a two-player submodular game can be cast as a supermodular game by reversing one of the player's order. The mentioned comparative statics outcome then follows by standard results on supermodular games (see the discussion above or the following footnote).

[23] And so need not necessarily reach a new fixed point. This is in contrast to the the order-preserving case, where the sequence $(s^\tau)_{\tau \in \mathbb{N}_0}$ is monotone and so converges on any compact set. If r is then continuous from below (and the

the direction suggested by intuition, the higher-order effects may turn these effects around and lead to counter-intuitive comparative statics outcomes even when the sequence converges to a new fixed point. Such occurrences cannot be ruled out in general.

Although comparative statics may yield counter-intuitive outcomes in general games with strategic substitutes, the situation is better in aggregative games. This is so because here we can focus on the backward reply correspondence which sometimes is much better behaved than the best-reply correspondence. Indeed, it is quite possible that best-replies are non-unique and non-convex while backward replies are singletons (cf. example 7 below).

For a parameterized family of games, the relevant backward reply correspondence (cf. section 2) is $B(Q,t) = \{s \in S : s_i \in R_i(h_i(Q, s_i), t), \text{ all } i \in \mathcal{I}\}$. If the game is quasi-submodular and every payoff function has increasing differences in (s_i, t) (for fixed s_{-i}), $R_i(g(s_{-i}, \theta_i), t)$ will be decreasing in s_{-i} and increasing in t in the strong set order. From this it is tempting to conclude by an argument along the lines of Milgrom and Roberts (1990), theorem 6, that B has greatest and least selections which are order-reversing in Q and order-preserving in t. This line of thinking is wrong, however, because for given Q and t the *domain* of $R_i(h_i(Q,\cdot),t)$ may not be a lattice (invalidating the use of Tarski's fixed point theorem).[24] The domain is however a semi-lattice, allowing us to conclude as follows:

Lemma 1 *Let Γ be a (quasi-)submodular game in a product order $\geq_S$ which is parameterized by $t \in T$ so that each of the payoff functions have increasing differences in $(s_i, t) \in S_i \times T$. Then the parameterized backward reply correspondence $B : X \times T \to 2^X \cup \emptyset$ has a least selection which is order-preserving in t and order-reversing in Q.*

Proof: *Let $\underline{r}_i$ denote the least selection from R_i. It follows easily from Topkis' theorem (Topkis (1978)) that $\underline{r}_i(h_i(Q, s_i), t)$ is order-preserving in s_i and t and order-reversing in Q. By a standard argument from Tarski's fixed point theorem, one shows that $H = \{s \in S : s_i \geq \underline{r}_i(h_i(Q, s_i), t) \text{ all } i\}$ is a directed complete semi-lattice. H is invariant under $\underline{r}_i$ and so has a least fixed point (the fixed point theorem used here can be found in e.g. Dugundji and Granas (2003)). This fixed point is also the least element of $B(Q,t)$, so B has a least selection. Since $\underline{r}_i$ is order-reversing in Q and order-preserving in t, so is the least fixed point selection.* □

Theorem 6 *Let Γ be a (quasi-)submodular game in a product order $\geq_S$ which is parameterized by $t \in T$ so that each of the payoff functions have increasing differences in $(s_i, t) \in S_i \times T$. Pick $t^1, t^2 \in T$ with $t^1 <_T t^2$ and take $s^1 \in \text{Fix}(t^1)$, $s^2 \in \text{Fix}(t^2)$. Then it cannot happen that $g(s^1) > g(s^2)$ if any one of the following two conditions is satisfied:*

1. *$B(Q,t)$ is strongly order-preserving in t.*
2. *$B(Q,t)$ is strongly order-reversing in Q.*

If B is an at most single-valued correspondence both 1. and 2. hold, and the conclusion consequently applies.

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least selection in a supermodular game *will* be continuous from below), the limit is a new fixed point. Obviously, this new fixed point will be higher than the original one, since so is every element in the sequence. The same argument can be used for the greatest selection (which is continuous from above) by considering instead a shock from t_2 to t_1 which produces a decreasing sequence.

[24] Take $h_i(Q, s_i) = Q - s_i$ and assume that $X \subset \mathbb{R}^N_+$. Even though $Q - s_i \geq 0$ and $Q - s_i' \geq 0$, it need not hold that $Q - s_i \vee s_i' \geq 0$.

Proof: *Assume to the contrary, that is pick* $s^1 \in B(Q^1, t^1)$, $s^2 \in B(Q^2, t^2)$ *such that* $g(s^1) = Q^1 > Q^2 = g(s^2)$. *Under 1.* $Z \geq Q^1 > Q^2 \geq Y$ *for all* $Z \in g \circ B(Q^1, t^2)$ *and some* $Y \in g \circ B(Q^1, t^2)$ *where the latter is true by lemma 1 combined with the argument used in the proof of the corollary to theorem 4. The proof in the second case is similar and is omitted.* □

If X is totally ordered, as is the case in one-dimensional games, the statement that $Q^1 > Q^2$ *cannot* hold of course means that $Q^1 \leq Q^2$ *must* hold. So in this case, theorem 6 yields strong comparative statics conclusions whether the shock is generalized or idiosyncratic. With idiosyncratic shocks, one can go further. The proof is simple: If Q increases, the backwards reply of any player not affected by the shock cannot increase. But then the backwards reply of the player affected by the shock must increase since $Q = g(s)$ where s is the PSNE (see Corchón (1994) for a similar argument).

Corollary *Consider a game* Γ *as in the theorem and assume that 2. holds (*B *is strongly order-reversing in* Q*),* X *totally ordered, and the shock idiosyncratic (*$\tilde{\pi}_i = \tilde{\pi}_i(s, t)$ *for some agent* i*, and* $\tilde{\pi}_j = \tilde{\pi}_j(s)$ *for all other agents* $j \neq i$*). Then* $t^2 \geq_T t^1$ *implies that* $s_i^2 \geq_{S_i} s_i^1$ *and* $s_j^2 \leq_{S_j} s_j^1$*, all* $j \neq i$ *for all* $s^1 \in \text{Fix}(t^1)$ *and* $s^2 \in \text{Fix}(t^2)$*.*

Another easily corollary is the following:

Corollary: *Consider a game* Γ *as in the theorem and let* $b(Q, t)$ *denote any selection from* $B(Q, t)$ *which is order-reversing in* Q *and order-preserving in* t *(such a selection exists by lemma 1). Then the conclusions of the theorem and its first corollary are valid for any two PSNE with* $Q^1 = b(Q^1, t^1)$ *and* $Q^2 = b(Q^2, t^2)$*,* $t^2 > t^1$*.*

Notice that to the extend that this second corollary is employed, it leads to *some* selection from the parameterized fixed-point correspondence $\text{Fix} : T \to 2^S$ with the predicted comparative statics properties. So among the selections of PSNE is one with "intuitive" comparative statics properties - but there may also be fixed point selections which are "counter-intuitive". Still, in many situations this is perhaps the best one can hope for.

Of course the previous results are of little use, unless one can find simple conditions under which the backwards reply correspondence is either strongly order-reversing in Q or strongly order-preserving in t. We shall not seek to exhaust this question here but leave it for future research. There is, however, a case which is worth mentioning because it shows that the previous results do improve upon existing results. If for every player i, the fixed point problem:

$$s_i \in R_i(h_i(Q, s_i), t) \tag{5}$$

has at most one solution given Q and t, then the backward reply correspondence will be single-valued. By lemma 1, this implies the conclusion of theorem 6 and its first corollary if X is totally ordered. Importantly, R_i need *not* be single-valued or even convex valued (the game quasi-concave) for this to hold. The next examples illustrate this point. The first is the text-book example of a problem which is not quasi-concave.

Example 7 (Cournot Equilibrium with U-Shaped Costs) *Consider a firm competing a la Cournot,* $\pi_i(s_i, s_{-i}; t) = s_i P^{t_1}(\sum_j s_j) - C_j^{t_2}(s_i)$ *where* S_i *is a compact subset of* $\mathbb{R}_+$*. Here* $t = (t_1, t_2)$ *are exogenous parameters affecting inverse demand and costs, respectively. Assume that* π_i *is smooth and that the firm's solution is interior (*$s_i > 0$*). Then* $B_i(Q, t) \subseteq \{s_i \in S_i : DC_i^{t_2}(s_i) = P^{t_1}(Q) + s_i DP^{t_1}(Q)\}$ *(the first-order condition is necessary for an interior optimum). It follows that* B_i *will be at most single-valued if the equation* $DC_i^{t_2}(s_i) = P^{t_1}(Q) + s_i DP^{t_1}(Q)$ *has at most one solution for all* *t and Q. The right-hand side is a linear function which intersects the second axis at $P^{t_1}(Q) > 0$ and has slope $DP^{t_1}(Q)$. The left-hand side is marginal costs. We assume that $DC^{t_2}(0) \leq P^{t_1}(Q)$ so the marginal cost of producing the first unit is below the market price. That costs are U-shaped of course means that marginal costs are first decreasing, $DC^{t_2}(s_i) \leq 0$ (increasing returns), and then begins to increase when s_i becomes sufficiently large, $DC^{t_2}(s_i) \leq 0$. It is easy to see graphically that under these conditions, there can be at most one solution to the equation; so B_i is at most single-valued.*

Example 8 (Cournot Equilibrium with Increasing Returns) *As in the previous example but now assuming that $DP^{t_1}(Q) - D^2C^{t_2}(s_i) \geq 0$ in which case $C_i^{t_2}$ must be concave (decreasing unit costs).*

Obviously there are numerous conditions which lead to the "'unique intersection"' outcome of the previous two examples ($(D^2C^{t_2}(s_i) - DP^{t_1}(Q))s_i + P^{t_1}(Q) - DC_i^{t_2}(s_i)$ non-negative or non-positive, etc.). In each case, the previous results apply. So, say, a shock to firm i's costs which decreases these on the margin will make aggregate output and the output of firm i increase, and the output of all other firms decrease. An excise tax in the sense of Vives (2000), section 4.3.1. (see discussion following theorem 5) will lower total output. And so forth. These results are as strong as one could possibly wish for.

Let us briefly turn to a case which is more restrictive than any of those mentioned, and relate this to some of the existing literature.[25]

With smooth payoff functions assume that $g(s) = \sum_i s_i$.[26] If the game is one of strict strategic substitutes, then by lemma 2 below every best-reply map will be a single-valued *contraction* if there exists $\alpha \in (0, 1)$ such that (here $\pi_i = \pi_i(s_i, x)$ where x is the realization of the aggregator):

$$D_{11}^2\pi_i + D_{12}^2\pi_i < \frac{(1-\alpha)}{\alpha}[D_{21}^2\pi_i + D_{22}^2\pi_i] \quad (6)$$

Of course, a contraction has exactly one fixed point, so backward replies will be single-valued.

Notice that in the smooth case, $D_{21}^2\pi_i + D_{22}^2\pi_i \leq 0$ if and only if $\tilde{\pi}_i = \tilde{\pi}_i(s_i, s_{-i})$ exhibits decreasing differences in $(s_i; s_{-i})$. If one assumes strictly decreasing differences: $D_{21}^2\pi_i + D_{22}^2\pi_i < 0$, then it is sufficient to assume that,

$$D_{11}^2\pi_i + D_{12}^2\pi_i < 0 \quad (7)$$

As explained in the next subsection, strictly decreasing differences and (7) together amount exactly to the "strong concavity" assumption of Corchón (1994). However, for a game to have strict strategic substitutes; it is not necessary to assume strictly decreasing differences (for example the strict dual single crossing property will do). Moreover, as discussed by Vives (2000) (p.98), even if the Corchón-type assumptions are taken as one's starting point; it is unnecessarily strong to assume that they should hold globally since all that matters (when stability is not considered) is that they hold on the graph of the best-reply correspondence.

3.3 Uniqueness, Global Stability, and Symmetry

To simplify the following results, we shall consider only the case where $\geq_S$ is the usual order, strategy sets are subsets of $\mathbb{R}^N$ and $g(s) = \sum_i s_i$. It is not too hard to generalize to arbitrary aggregators and

[25] Here we consider general payoff functions. The multivalued generalization can be found in the next subsection.

[26] As will be explained in the next subsection both smoothness and the linear form of the aggregator are for simplicity reasons and can be dispensed with.

arbitrary vector orders (whose interior must be non-empty) - but it is somewhat cumbersome and so we shall leave this for the interested reader to pursue.

Let $\pi_i(q_i, q_j)$ denote the player's payoff function where q_j is the realization of the aggregator (for example in the Cournot model $\pi_i(q_i, q_j) = q_i P(q_j) - c_i(q_i)$). Assuming that π_i is smooth in a neighborhood containing $S_i \times X$, let $D^2_{ij}\pi_i(q_i, q_j)$ denote the N by N matrix of second-order partial derivatives.

Definition 4 *π_i is said to satisfy the GV condition (GV stands for Granot-Veinott) if there exists a real, non-zero, and non-negative N by N matrix $\mathbf{A}$ such that for all $(q_i, q_j) \in S_i \times X$:*

$$(D^2_{11}\pi_i + D^2_{12}\pi_i)^T \mathbf{A} \ll (D^2_{12}\pi_i + D^2_{22}\pi_i)^T (\mathbf{I} - \mathbf{A}) \tag{8}$$

If (8) holds with (weak) inequality at the diagonal (but not necessarily off the diagonal), then we say that π_i satisfies the (weak) GV condition at the diagonal.

Though (8) may look somewhat frightening, it is just the second-order condition for strictly increasing differences in (y, z) in $\pi_i(z - \mathbf{A}y, z + (\mathbf{I} - \mathbf{A})y)$. This, in turn, is just player i's payoff function, $\pi_i(s_i, s_i + y)$, after the substitution $z = \mathbf{A}y + s_i$ (here $y \equiv \sum_{j \neq i} s_j$). Of course, the previous smoothness assumptions are unnecessary for expressing this condition and can therefore easily be dispensed with. We remark that the GV condition is very closely related to the notion of *double subadditivity* introduced by Granot and Veinott (1985). Hence the term Granot-Veinott condition.[27]

Of course the GV condition coincides with the GV condition at the diagonal if $N = 1$. Then $\mathbf{A}$ is just a positive scalar, $\mathbf{A} = \alpha > 0$ say, and the GV condition reduces to: There exists $\alpha > 0$ such that for all $(q_i, q_j) \in S_i \times X$:

$$D^2_{11}\pi_i + D^2_{12}\pi_i < \frac{(1-\alpha)}{\alpha}[D^2_{21}\pi_i + D^2_{22}\pi_i] \tag{9}$$

For example in the Cournot model with $N = 1$ where $\pi_i(q_i, q_j) = q_i P(q_j) - C_i(q_i)$, (9) will be satisfied with $\alpha = 1$ provided that $DP - D^2 C_i < 0$ (if C_i is strictly convex and P is decreasing this will hold).[28] Note that (9) with $\alpha \leq 1$, implies that the player's payoff function $\pi_i(s_i, s_i + \sum_{j \neq i} s_j)$ is strictly concave in s_i if the game is submodular. In fact, this implies "strong concavity" in the sense of Corchón (1994): $D^2_{11}\pi_i + D^2_{12}\pi_i < 0$ and $D^2_{21}\pi_i + D^2_{22}\pi_i \leq 0$.[29]

In the following, the (non-negative) matrix $\mathbf{A}$ is equipped with the norm: $\|\mathbf{A}\|_\infty = \max_n \sum_{k=1}^N A_{nk}$ (the maximum absolute row sum norm). So for example, $\|\mathbf{I}_N\|_\infty = 1$ where $\mathbf{I}_N$ is the N by N identity matrix.

[27] Curtat (1996) (who calls double subadditivity, doubly increasing differences) establishes Lipschitz continuity of extremum selections in a supermodular optimization problem (Curtat (1996), theorem 2.3.). Curtat's approach can be transfered to the present framework by replacing the term $\mathbf{A}y$ in the proof of lemma 2 with a term of the form $(\phi(y), \ldots, \phi(y))$ where ϕ is a Lipschitz continuous order-preserving real valued function. Note that this is a special case of the GV condition if $\phi(y)$ is linear. Of course, the most general approach would simply replace $\mathbf{A}y$ with an order-preserving and Lipschitz continuous vector valued function; but the gain from doing so is doubtable since, in applications, one faces the problem of actually finding such a function.

[28] Amir and Lambson (2000) provide an example where C_i is in fact globally concave, but where it never-the-less holds that $DP - D^2 C_i < 0$.

[29] Corchón (1994) actually assumes that the inequality is strict in each of these (which, when π_i is C^2 on a compact set, implies that the GV condition holds for some $\alpha < 1$). The condition $D^2_{21}\pi_i + D^2_{22}\pi_i \leq 0$ is decreasing differences. Taken together, the two conditions imply that $D^2_{11}\pi_i + D^2_{12}\pi_i + D^2_{21}\pi_i + D^2_{22}\pi_i < 0$ (which is strict concavity of π_i in the player's strategy). Incidentally, strict concavity is just (9) in the limit $\alpha \to \infty$.

Lemma 2 *Let Γ be a game with strategic substitutes (in the usual order), and aggregative with $g(s) = \sum_i s_i$. Furthermore assume that S_i is a sublattice of $\mathbb{R}^N$ for each i. If the payoff function of player i satisfies the GV condition for the matrix $\mathbf{A}$, then player i's best-reply correspondence has a Lipschitz continuous selection $r_i : X \to S_i$, $X, S_i \subset \mathbb{R}^N$, with Lipschitz constant smaller than or equal to $\|\mathbf{A}\|_\infty$. If $\|\mathbf{A}\|_\infty = 1$, such a selection is consequently non-expansive, and if $\|\mathbf{A}\|_\infty < 1$, it is a contraction. If π_i satisfies the GV condition at the diagonal for some $\mathbf{A}$, then player i's best-reply correspondence has a selection r_i which is Lipschitz continuous "at the diagonal", i.e., for all n, $r_i^n(x^1, \ldots, x^n, \ldots, x^N)$ will be Lipschitz in x^n.*

Remark: *As is clear from the proof, the result is also valid if the GV condition holds only with weak inequalities, provided that the game is strictly submodular.*

Proof: *Let $Z^*(y) = arg\max_{z \geq \mathbf{A}y} \pi_i(z - \mathbf{A}y, z + (\mathbf{I} - \mathbf{A})y)$. Note that $R_i(y) = Z^*(y) - \mathbf{A}y$ where $R_i(y) = arg\max_{x \geq 0} \pi_i(x, x+y)$ (here 0 denotes the bottom element of S_i). Since the game has strategic substitutes, R_i has an order-reversing selection, which is called x^*, and the corresponding selection from Z^* is denoted z^*. The constraint correspondence $\Gamma(y) = \{v : v \geq \mathbf{A}y\}$ is ascending because $\mathbf{A}$ is non-negative and so $\mathbf{A}$ maps $\mathbb{R}_+^N$ into $\mathbb{R}_+^N$ (which is the positive cone of the usual order). The GV condition thus implies that the objective of Z^* has strictly increasing differences, so for the selection x^* and all $y' \geq y$, $x^*(y') + \mathbf{A}y' \geq x^*(y) + \mathbf{A}y$, or equivalently, $x^*(y') - x^*(y) \geq \mathbf{A}(y - y')$. Since x^* is order-reversing, this implies in turn that for all $y \in \bar{S}_{-i} := \{y = \sum_{j \neq i} s_j \in \mathbb{R}^N : s_j \in S_j \text{ for all } j\}$, and all $d \geq 0$ with $y + d, y - d \in \bar{S}_{-i}$: $0 \geq x^*(y + d) - x^*(y) \geq -\mathbf{A}d$ and $0 \leq x^*(y - d) - x^*(y) \leq \mathbf{A}d$. For fixed d, x^* is consequently "directionally Lipschitz" in the sense that $\|x^*(y+d\epsilon) - x^*(y)\| \leq \|\mathbf{A}d\|\epsilon$, for all $\epsilon \in \mathbb{R}$ such that $y + \epsilon d, y - \epsilon d \in \bar{S}_{-i}$. Pick $d \gg 0$ such that $d \geq \frac{y-y'}{\|y-y'\|} \geq -d$ for all $y, y' \in \bar{S}_{-i}$ and let $\alpha := \|\mathbf{A}d\|$. For $y, y' \in \bar{S}_{-i}$, let $u = y' - y$ and note that when $\delta = \|u\|$, $\delta d \geq u \geq -\delta d$ (since $d \geq \frac{u}{\|u\|} \geq -d$). But then $f(y - \delta d) - f(y) \geq f(y + u) - f(y) \geq f(y + \delta d) - f(y)$, which implies, $\|f(y') - f(y)\| = \|f(y+u) - f(y)\| \leq \max\{\|f(y - \delta d) - f(y)\|, \|f(y + \delta d) - f(y)\|\} \leq \alpha\delta = \alpha\|y' - y\|$, where the last inequality was shown to hold above. Let $D = \{d \geq 0 : d \geq \frac{y'-y}{\|y'-y\|} \geq -d \text{ for all } y', y \in \bar{S}_{-i}\}$ and define $\rho = \min_{d \in D} \|\mathbf{A}d\|$. The previous argument implies that x^* is (globally) Lipschitz continuous with Lipschitz constant less than or equal to ρ. If, $\|\cdot\|$ is the maximum norm, then $\rho \leq \|\mathbf{A}(1, 1, \ldots, 1)^T\| = \|\mathbf{A}\|_\infty$, from which the first statements of the theorem follow. The statement concerning the GV condition at the diagonal is shown by applying the exact same argument to the n'th coordinate of the order-reversing selection holding x^{-n} fixed.* □

Say that s^* is a *globally stable* fixed point if, given an arbitrary initial strategy $s^0 \in S$, the best-reply iteration $s^t = R(s^{t-1})$, $t = 1, 2, 3, \ldots$ converges to s^* in the usual topology. Recall that a game has strict strategic substitutes if every best-reply map is strongly order-reversing.

Theorem 7 (Uniqueness and global stability theorem) *Let Γ be a game with strict strategic substitutes in the usual order and aggregative with $g(s) = \sum_i s_i$, $S_i \subset \mathbb{R}^N$. If for every i, π_i satisfies the GV condition* (8) *for some $\mathbf{A}$ with $\|\mathbf{A}\|_\infty < 1$ (where $\mathbf{A}$ may depend upon i), there exists a unique and globally stable PSNE. If the game is symmetric, this PSNE will be a symmetric PSNE.*

Proof: *If every selection from a best-reply correspondence is order-reversing and continuous (and under the conditions of the theorem this will be the case), then this correspondence must be single-valued. From this, lemma 2, and Banach's contraction principle the first statement is immediate. The second (concerning symmetric games) follows by the argument used in the next theorem.* □

Example 9 *Let $N = 1$ and take $\pi_i(q_i, q_j) = q_i P(q_j) - C_i(q_i)$ (the Cournot model), $S_i = [0, b_i]$, $b_i \in \mathbb{R}_+$, and assume that for every i, $0 < q_i D^2 P(q_j) + D^2 C_i$. As is seen, this is (9) with $\alpha = 0.5$,* *hence every best-reply map is a contractive function and there will therefore exist a unique and globally stable PSNE.*

Theorem 7 (which might more rightfully be seen as a corollary to lemma 2), substantially strengthens the conclusion of our main theorem 1 under the conditions of the GV condition. If, for player i, the GV condition holds for some $\mathbf{A}$ but *not* one for which $\|\mathbf{A}\|_\infty < \mathbf{1}$, then a selection will still be (Lipschitz) continuous by theorem 2. The existence of a fixed point then follows from Brouwer's fixed point theorem if S is convex and compact, but our main theorem implies the existence of a PSNE under much weaker conditions.

Symmetric games with strategic substitutes *do not* necessarily posses symmetric PSNE. Even if the game is aggregative and has only two players (*hence* can be cast as a *supermodular* game with a change of order, cf. Vives (1990)), it may fail to have a symmetric PSNE. Thus, take $I = 2$, $S_i = S_j \subset \mathbb{R}$, and $g(s) = s_1 + s_2$. Then $(s^*, s^*) \in S_i \times S_i$ is a symmetric PSNE if and only if it solves $s^* \in R(s^*)$. But this problem need not have a solution:

Example 10 *Let $I = 2$, $S_i = S_j = \{0, 1\}$, which is a chain, and $R(0) = \{1\}$, $R(1) = \{0\}$, which is an order-reversing singleton correspondence (hence strongly order-reversing). There clearly does not exist a symmetric PSNE. On the other hand there do exist PSNE (theorem 1) and the set of PSNE forms an antichain (theorem 3). Indeed $(s_1^*, s_2^*) = \{0, 1\}$ and its permutation $(s_1^*, s_2^*) = \{1, 0\}$ are both fixed points of the joint best-reply correspondence.*

In comparison, a symmetric game with strategic complementarities has a symmetric PSNE (this statement does not include Cournot duopoly since this is not supermodular if agent's strategy sets are endowed with the same order). The next result addresses the problem thus raised. In the single market Cournot model, it is well known that if costs are convex and inverse demand decreasing, then a symmetric PSNE exists (cf. Roberts and Sonnenschein (1976)). As noticed by Amir and Lambson (2000), this result immediately extends to the case where $DP - D^2C_i < 0$ (which is (9) when $\alpha = 1$). In fact, the result readily generalizes to the general one-dimensional model assuming merely that the GV condition (9) holds for some $\alpha > 0$. The reason is in any of these cases that the best-reply map, which maps from $\mathbb{R}$ into subsets of $\mathbb{R}$, has a selection which "has no jumps down". Theorem 8, which relies on theorem 1 and therefore is a new result, shows that in submodular games with strategy sets of arbitrary dimension, symmetric PSNE exist provided that there are "no jumps down" at the *diagonal* of the best-reply correspondence:

Theorem 8 (Existence of Symmetric PSNE) *Let Γ be a symmetric submodular game (in the usual order) and aggregative with $g(s) = \sum_i s_i$, $S_i \subset \mathbb{R}^N$. If for every i, π_i satisfies the GV condition at the diagonal (with no restriction on $\|\mathbf{A}\|_\infty$) and if the strategy set is convex, there exists a symmetric PSNE. The same result holds if Γ is strictly submodular and π_i satisfy the weak GV condition at the diagonal. Furthermore, if each S_i is a chain and the game is either regular or strictly submodular, the symmetric PSNE is unique.*

Proof: *Clearly $(s^*, \ldots, s^*)$, $s^* \in S_i$, is a symmetric PSNE if and only if $s^* = r((I-1)s^*)$, where r is some selection from R. By lemma 2, $r^n((I-1)(s^{-n}, s^n))$ is (Lipschitz) continuous in s^n given $s^{-n} = (s^1, \ldots, s^{n-1}, s^{n+1}, \ldots, s^N)$. Define $p^n(s^{-n}) = \{\bar{s}^n : \bar{s}^n = r^n((I-1)(s^{-n}, \bar{s}^n))\}$. By Brouwer's fixed point theorem, p^n will be non-empty valued, and moreover it will be a function (a singleton correspondence) because r^n is order-reversing in s^n (a decreasing function from $\mathbb{R}$ to $\mathbb{R}$ can intersect the 45°-line at most once). Finally, p^n will clearly be order-reversing. $s^n = p^n(s^{-n})$, all n, if and only if $s^n = r((I-1)(s^{-n}, s^n))$ for all n. But the function $p = (p^n)_{n=1}^N$ satisfies all conditions of* *theorem 1, so a fixed point exists. Uniqueness follows from theorem 2 in case the game is strictly submodular, and from theorem 3 if it is regular. Indeed, if there existed two symmetric PSNE these would, because of symmetry, necessarily be ordered by $\geq_S$ which would be in contradiction with these theorems' conclusion.* □

The previous theorem shows, among other things, that if one studies symmetric two-player games with strategic substitutes; it is not advantageous to recast these as a supermodular game if attention is focused on symmetric equilibria. The existence of symmetric PSNE is much more effectively addressed in the submodular framework. We mention again that extending the above results to arbitrary aggregators is straight-forward, albeit cumbersome (one "simply" inserts the expression for the aggregator and derives the conditions for increasing differences as in the proof of theorem 2).

4 Examples of Games with Strategic Substitutes

In this section we present several examples of aggregative games with strategic substitutes, and use these to illustrate the results and methods of the preceding sections.

4.1 Multimarket Cournot Equilibrium

Existence of a pure strategy Cournot equilibrium is a topic which has been subject to much interest. In symmetric one-good, convex-cost Cournot games, existence was proved for any number of firms by McManus (1964) and Roberts and Sonnenschein (1976). The latter two authors (Roberts and Sonnenschein (1977)) pointed out that in general, a PSNE need not exist in such games. On the other hand, the setting of Roberts and Sonnenschein (1977) do admit PSNE if there are strategic substitutes, because there are only two firms (so Tarski's fixed point theorem can be made to bear, see *e.g.*, Vives (1990)). With any number of firms, the presence of general equilibrium effects typically destroys additivity of the aggregator (see also the study of price competition a la Bertrand below) and as explained by Amir (1996), Section 3, such games tend towards strategic substitutes when there are (non-decreasing) costs. Using theorem 1 we are able - for the first time in the literature when there are more than two firms - to conclude that there exists a pure strategy Nash equilibrium if the game has strategic substitutes. Using the results of section 3 we are, moreover, able to characterize such games from various perspectives. The specific model we consider is closely related to Bulow et al. (1985), who in their seminal study consider multimarket Cournot duopolies and introduce the distinction between strategic substitutes and strategic complements. In the strategic complementarities case, Topkis (1998), section 4.4.3., generalizes this to $N \in \mathbb{N}$ markets and $I \in \mathbb{N}$ players. It is this model we consider here with strategic substitutes.

Consider a set $\mathcal{I} = \{1, \ldots, I\}$, $I \geq 2$ of firms who compete in $N \geq 2$ markets. A strategy for firm $i \in \mathcal{I}$, is a vector $s_i = (s_i^1, \ldots, s_i^N) \in S_i = [0, x_i] \subset \mathbb{R}_+^N$ where $x_i \gg 0$. The payoff to firm i, its profit function, is assumed to be an upper semi-continuous function of the form:

$$\pi_i(s_i, s_{-i}) = s_i^1 P^1(g(s)) + \ldots + s_i^N P^N(g(s)) - C_i(s_i) \tag{10}$$

where $g : \prod_i S_i \to X \subseteq \mathbb{R}_+^N$ is an aggregator. In the simplest situation, inverse demand in market n, P^n, depends only on the total supply of good n in which case we take $g(s) = \sum_i s_i$ and $P^n(g(s)) = \tilde{P}^n(g_1(s)) = \tilde{P}^n(\sum_{i=1}^I s_i^n)$.[30] In the more general case, demand in market n depends on total supply in all markets, so g is as before and $P^n(g(s)) = \tilde{P}^n(\sum_{i=1}^I s_i)$. We could also assume that some firm holds a monopoly in one of the markets, and compete in the remaining markets (we do so simply

[30] Note that this will be the case in repeated Cournot games.

by restricting the strategy sets). This would result in the model studied in the two-firm, two-market case by Bulow et al. (1985).

Strategic substitutes in this model means that if a firm competes more aggressively in a market, then this will make the other firms lower their supply in that market. By Topkis' monotonicity theorem (Topkis (1978)), this outcome is ensured if each S_i is a lattice, (10) is supermodular in s_i, and has decreasing differences in $(s_i; s_{-i})$. A pure strategy Nash equilibrium will then exist irrespective of any convexity or quasi-concavity assumptions.

If every firm chooses according to a decreasing selection from their best-reply correspondence, we have by theorem 2 that the set of pure strategy Nash equilibria is an antichain. If the game is one of strict strategic substitutes, two ordered PSNE can only differ in the entry of a single firm (corollary to theorem 3). By theorem 4, when parameterizing such that best-replies are increasing in the exogenous parameter, an increase in this parameter cannot lead to a situation where all outputs decrease. In fact, in symmetric equilibrium, we know that the PSNE is order-preserving in the exogenous variable (theorem 5).

Under stronger conditions we can do better. Thus take $P^n(g(s)) = \tilde{P}^n(\sum_{i=1}^{I} s_i^n)$, in which case the game is submodular if each C_i is submodular and $\tilde{P}^n$ is non-increasing and concave.[31] If the game is symmetric, a symmetric PSNE exists by theorem 8 if, in addition, $C(s) = C(s_1, \ldots, s_N)$ is convex in each of its coordinates (since this is the GV condition at the diagonal with $\mathbf{A}$ equal to the identity matrix). If the game is strictly submodular (*e.g.* if each $\tilde{P}^n$ is strictly decreasing), this symmetric PSNE is unique. If, furthermore, C is a strictly convex function, the symmetric PSNE is the unique PSNE and it is globally stable (theorem 7). In this case, the backward reply correspondence is single-valued and so theorem 6 as well as its corollaries apply, leading to a number of stronger comparative statics conclusions for generalized as well as idiosyncratic shocks.

4.2 Team Projects

Whereas in models of imperfect competition players' payoff typically decrease when other players raise their strategy, there are also economic situations where the opposite is true. An example is in the team project game studied by Dubey et al. (2002). Here agents contribute effort to a common project which may or may not succeed, but is more likely to do so the more effort each agent contributes. Strategic substitutes arise because the team's members have an incentive to 'free-ride', *i.e.*, contribute less if they observe that other agents contribute more. To take a familiar example, consider $I = 3$ economists who cowrite an article (so success=publication, and effort=time devoted to writing the paper). Each player has strategy set $S_i = [0, 1]$, the interpretation being that effort is equal to probability of success $s_i \in S_i$ for player i. Consider then the aggregator $g(s) = 1 - (1 - s_1)(1 - s_2)(1 - s_3)$, which is the probability of success of a project in the situation where every agent has to fail for the project to fail (it is sufficient that one of the authors proves a good theorem). The function g is strongly increasing, and to be sure it is an aggregator: If agent i knows the probability of publication $Q = g(s_{-i}, \theta_i)$ when she does not work ($\theta_i = 0$), she is able to correctly anticipate the probability of publication, $g(s)$, for all $s_i \in S_i$ via the function $F_i(Q, s_i) = Q + s_i - Qs_i$ (compare with (2)).

The aggregator g has decreasing differences, hence we face a game with strategic substitutes if payoff functions are of the form $\pi_i(s) = P_i(g(s)) - C_i(s_i)$, where C_i is an arbitrary 'effort cost function', and P_i is an increasing, concave function which measures the 'return to success'.[32] Dubey

[31] Of course concavity is not necessary for this outcome: The necessary and sufficient condition for decreasing differences in case P^n is smooth reads: $DP^n + s_i^n D^2 P^n \leq 0$ all n, i and $s \in S$. On top of this, we could look for conditions such that (10) becomes a (weakly) quasi-submodular game, which would yield still weaker conditions.

[32] As everywhere else in this paper concavity of P_i is unnecessary although highly practical in this specific case (if f is concave and g has decreasing differences then $f \circ g$ has decreasing differences, cf. Topkis (1978), section 3). In the smooths case there will be decreasing differences to player i, iff $-(1 - s_j)DP + (1 - s_k)(1 - s_j)D^2 P \leq 0$, where $j, k \neq i$.

et al. (2002) prove the existence of a PSNE in this game. Using the results in this paper a number of new results and generalizations are possible.

As for generalizations, the aggregator does not have to be of the specific form of Dubey et al. (2002). It might for example be more realistic to assume that the probability of success is a function of effort, *i.e.*, $p_i = h_i(s_i)$, where p_i is now the probability that agent i succeeds when his effort is s_i, and h_i is a function from S_i into $[0, 1]$ (not necessarily onto, so it may be that an agent cannot be sure of success no matter what his effort is). Under this description we get $g(s) = 1 - (1 - h_1(s_1))(1 - h_2(s_2))(1 - h_3(s_3))$, which is an aggregator provided that each h_i is strictly increasing. Again this aggregator has decreasing differences so a PSNE will exist. Another generalization in the one-dimensional case would be to consider other probability distributions. Finally, this paper's existence result applies also to the case where the team has $N \geq 2$ projects. One would, for example, then take $g(s) = (1 - (1 - s_1^n)(1 - s_2^n)(1 - s_3^n))_{n=1}^N$, *i.e.*, the vector of probabilities of success for each of the N projects. The effort cost function should be a submodular function $C_i = C_i(s_i^1, \ldots, s_i^N)$, and the payoff function could be the sum of increasing and concave 'individual project payoff functions similar to in the multimarket oligopoly model studied in a previous subsection.

Now, existence is of course just a consistency check. What else can we conclude ? First, if the game is strictly submodular; the set of equilibria is an antichain except possibly for ordered PSNEs where at most one of the authors works less than he could have done. Now, the "culprit" will necessarily be indifferent between working more or less (both are best-replies given the other two authors' effort). Intuitively, he is aware that working less decreases the probability of publication. Working more increases the success of publication but also increases effort-cost and these two effects exactly balance each other in terms of the author's payoff. If we are willing to assume that in such a situation authors are bound by a selection rule (*e.g.*, "if indifferent, pick the greatest best-reply") the equilibrium set will become an antichain (theorem 2). For those of us who coauthor papers this has an important implication: There will be no coordination failure (although the group's social optimum is of course not necessarily reached through non-cooperative behavior). In the one-dimensional case, if the game is symmetric there is at most one symmetric equilibrium (corollary to theorem 3). And by theorem 8 a symmetric PSNE will exist if, for example, costs are convex (see the previous example for a similar argument). As in the other examples studied in this section one can also ask and answer various comparative statics questions. As an example, assume that the team is symmetric and take the case where the probability of success is a function of effort; but now also of an exogenous parameter t: $p_i = h(s_i, t)$. If h is increasing in t, so t measures how well the team knows the editor, say; an increase in t will make all of the team's members place *less* effort into the project in symmetric equilibrium. And if that is not a realistic prediction, what is ?

4.3 Bertrand Oligopoly with Differentiated Products

Let each strategy set, S_i, be a compact subset of $\mathbb{R}_+$. The profit function of firm i is $\tilde{\pi}(p_i, p_{-i}) = p_i D_i(p) - C_i(D_i(p))$, where $p_i \in S_i$ is the price set by firm i, $p = (p_1, \ldots, p_I) \in S$, and $D_i : S \to \mathbb{R}_+$ is the demand function for good i (S is the joint strategy set). If this game is (quasi-)supermodular, it is known that a PSNE exists (see *e.g.* Vives (2000)). If it is not, existing literature has no answer to the existence question without quasi-concavity.[33] In addition, comparative statics of fixed points is typically a daunting task even if sufficient assumptions are imposed for the implicit function theorem

[33] Milgrom and Roberts (1990) (see also Topkis (1979)), study Bertrand oligopoly as a log-supermodular game under the assumption of linear costs where $\tilde{\pi}(p_i, p_{-i}) = (p_i - c_i) D_i(p)$. The game is then log-supermodular if $\frac{\partial^2 \log D_i(p)}{\partial p_i \partial p_j} \geq 0$ for all i, j. With linear costs, the game is similarly log-submodular, and so a game of strategic substitutes, if $\frac{\partial^2 \log D_i(p)}{\partial p_i \partial p_j} \leq 0$ for all i, j. This outcome comes about, for example, if goods are gross complements and the demand function exhibits decreasing differences.

to apply. No doubt, comparative statics runs most smoothly in the case of strategic complementaries. But as we shall see, in many instances this paper's results can be used with success also when there are strategic substitutes.

To keep things simple, assume that demand is of the constant expenditure form, $D_i(p) = \frac{1}{p_i}\frac{h_i(p_i)}{\sum_j h_j(p_j)}$ where each $h_j : S_j \to \mathbb{R}_{++}$ is an increasing function (so goods are gross complements). As is easily verified $g(p) = \sum_j h_j(p_j)$ is an aggregator (it is separable and any separable function is an aggregator as shown in example 2). Since we can write $\tilde{\pi}_i(p_i, p_{-i}) = \pi(p_i, g(p))$ where $\pi(p_i, g(p)) = \frac{h_i(p_i)}{g(p)} - C_i(\frac{1}{p_i}\frac{h_i(p_i)}{g(p)})$, this game is aggregative, as indeed it will be under a great many other demand specifications. Specific examples which belong to the constant expenditure form are CES with complementarities and (increasing) exponential demand (see Vives (2000), chapter 6). Note that we *do not* assume that demand is symmetric across the goods ($h_i = h$ all i for some function h), so the aggregator will (generally) not be a symmetric function either. Under these, quite general, conditions we now have by theorem 1 that there will exist a PSNE.

Assuming that all firms produce in equilibrium (so solutions are interior) and that D_i is differentiable; the following first-order conditions are necessary (but need not be sufficient):

$$\frac{p_i - DC_i}{p_i} = \frac{1}{\eta_i(p)}, \text{ for all } i \in \mathcal{I} \tag{11}$$

where $\eta_i(p)$ is the elasticity of demand. In the constant expenditure case we may write η_i as a function of p_i and Q and insert to get the following *necessary* condition for $p_i \in B_i(Q)$ (here, as throughout, B_i is the backward reply correspondence):

$$1 + \frac{Dh_i(p_i)}{Q} - \frac{p_i Dh_i(p_i)}{h_i(p_i)} = \frac{p_i}{p_i - DC_i(\frac{h_i(p_i)}{p_i Q})} \tag{12}$$

If we impose assumptions such that this equation has at most one solution for every Q we can, because strategy sets are chains, apply the first corollary to theorem 6.[34] Thus take $D_i = D_i(p, t)$ and increasing in t (a demand shock). Then in a PSNE, p_i will be increasing in t and p_j, $j \neq i$, will be decreasing in t. If the game is symmetric, things are simpler. With parameterized demand $D(p; t) = D_i(p, t)$ all i and increasing in t, a positive demand shock will unambiguously lead to a price increase in symmetric equilibrium (theorem 5). And, regardless of whether the game is symmetric or not, such a positive shock to demand could never lead to a new PSNE where all firms lower their price (theorem 4). Also worth mentioning is that, in contrast to the supermodular case, where the presence of multiple equilibria would motivate government intervention so as to "push" the firms to the least PSNE (where prices are at their minimum), the equilibrium set will now tend towards being an antichain. If so, any change from one PSNE to another will be associated with a rise in some prices and a fall in others. Consequently, government intervention in the sense of "coordination requests" will generally *not* lead to a Pareto improvement for the consumers.

4.4 Competition Between Teams

In all of the previous models, agents were implicitly assumed to have the *same* strategic attitudes toward each other ("if you get more aggressive, I do to", "if you increase your effort, I will lower mine", etc., etc.). It is not hard, however, to think of games where a *mixture* of the two coexist. Indeed, whenever a team or coalition with common interests face one or more other teams with opposing

[34] There is a variety of cases where B_i will be at most single-valued. To mention one, if $h_i(p_i) = p_i^{\epsilon_i}$, $0 < \epsilon_i < 1$; this will hold provided that the term $\frac{DC_i(p_i^{\epsilon_i - 1} Q^{-1})}{p_i}$ is decreasing in p_i.

interests, this will be the case. Below we consider an arms race with an alliance on one side. A similar example is the game studied experimentally by Dufwenberg and Gneezy (2000). There the teams are groups of firms which compete monopolistically with each other and share profits within groups according to a simple bargaining scheme. If the members of a team have an incentive to "free-ride", this will, as we shall see, give rise to a game of strategic substitutes.

Let us first consider the general case in one dimension ($S_i \subset \mathbb{R}$ all i). Let $\tilde{\pi}(s_i, s_{-i})$ denote the actual payoff function of player i. Assume that each $\tilde{\pi}_i$ is twice continuously differentiable, and to simplify notation let $\tilde{\pi}(s, \bar{s}) = \sum_{i=1}^{I} \tilde{\pi}_i(s_i, \bar{s}_{-i})$. Then consider the matrix of cross-derivatives, $D^2_{s\bar{s}}\tilde{\pi}$, which is the matrix with $\frac{\partial^2 \tilde{\pi}}{\partial s_i \partial \bar{s}_j}$ in the i'th row and j'th column. $D^2_{s\bar{s}}\tilde{\pi}$ will have a zero diagonal, so if $D^2_{s\bar{s}}\tilde{\pi}$ is non-positive (non-negative), the game is submodular (supermodular) *in the usual order*. The point here is that if the game consists of a mixture of agents' attitudes, $D^2_{s\bar{s}}\tilde{\pi}$ might for example look like:

$$(13) \qquad D^2_{s\bar{s}}\tilde{\pi} = \begin{bmatrix} 0 & - & + \\ - & 0 & - \\ + & - & 0 \end{bmatrix} \text{ or } D^2_{s\bar{s}}\tilde{\pi} = \begin{bmatrix} 0 & + & - \\ + & 0 & + \\ - & + & 0 \end{bmatrix}$$

More generally, consider a game with I-players and assume that $D^2_{s\bar{s}}\tilde{\pi}$ after suitable symmetric permutations of rows and columns has the following combinatorial sign structure:[35]

$$(14) \qquad D^2_{s\bar{s}}\tilde{\pi} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

where $A_{11} \in \mathbf{R}^{M\times M}$ and $A_{22} \in \mathbf{R}^{(I-M)\times(I-M)}$ are non-positive, and $A_{12} \in \mathbf{R}^{M\times(I-M)}$ and $A_{21} \in \mathbf{R}^{(I-M)\times I}$ are non-negative matrices. As may be shown, this sign-combinatorial structure is *necessary and sufficient* for the game to be submodular with respect to an order whose positive cone is an orthant.[36] Similarly, the game will be supermodular after such a change of order if and only if symmetric permutations can bring $D^2_{s\bar{s}}\tilde{\pi}$ to the form (14) where A_{11} and A_{22} are non-negative, and A_{12} and A_{21} are non-positive.[37] So in (13) we conclude that the first cross-derivative matrix reflects a game with strategic complementarities while the second underlies a game with strategic substitutes.

Now to the concrete example of an arms race. The structure is a one-shot race as studied by Milgrom and Roberts (1990), but with three countries, two of which are allied. Call the countries the US, the UK, and the Soviet Union (SU). The strategy sets are of the form $S_i = [0, b_i] \subset \mathbb{R}$, $b_i > 0$. The payoff to the USA and the UK - who form an alliance - is $\tilde{\pi}_i(s_{\mathrm{US}}, s_{\mathrm{UK}}, s_{\mathrm{SU}}) = B_i(s_{\mathrm{US}} + s_{\mathrm{UK}} - s_{\mathrm{SU}}) - C_i(s_i)$, $i = \mathrm{US}, \mathrm{UK}$, where B_i are smooth and concave functions and C_i are arbitrary functions. The payoff to the USSR is $\tilde{\pi}_{\mathrm{SU}}(s_{\mathrm{US}}, s_{\mathrm{UK}}, s_{\mathrm{SU}}) = B_{SU}(s_{\mathrm{SU}} - s_{\mathrm{US}} - s_{\mathrm{UK}}) - C_{\mathrm{SU}}(s_{\mathrm{SU}})$, with the same conditions on B_{SU} and C_{SU} as for B_i, C_i. Calculating the second-order cross-derivative matrix $D^2_{s\bar{s}}\tilde{\pi}$, we get the second combinatorial sign structure in (13) by placing the US in the first row/column, the Soviet Union in the second row/column, and the UK in the third row/column. So this is a game of strategic substitutes in the order $\geq_S$ defined by $s^1 \geq_s s^2 \Leftrightarrow s^1_{\mathrm{US}} \geq s^2_{\mathrm{US}}$, $s^1_{\mathrm{SU}} \leq s^2_{\mathrm{SU}}$, $s^1_{\mathrm{UK}} \geq s^2_{\mathrm{UK}}$ (whose positive cone is $H = \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+$). As is obvious, the game is also aggregative with aggregator $g(s) = s_{\mathrm{US}} + s_{\mathrm{UK}} - s_{\mathrm{SU}}$ (which preserves the order $\geq_S$).

[35] Symmetric permutation means that if row i and j are interchanged then column i and j must be interchanged also.

[36] Let H be a pointed, convex cone in $\mathbb{R}^N$ (a set if pointed if $a, -a \in H$ implies $a = 0$). Such a cone defines a (partial vector) order by virtue of $x \geq y \Leftrightarrow x - y \in H$. The order is then said to have positive cone H. H is an orthant if it is a set of the form $H = \prod_n \mathbb{R}_{\alpha(n)}$, $\alpha(n) \in \{-, +\}$ all n. For example $\mathbb{R}^N_+$ is an orthant, and it is the positive cone of the usual order. For an in depth treatment of such orders and their relationship to monotone comparative statics the reader is referred to Jensen (2004).

[37] These results are non-trivial. For further elaboration and references see Smith (1988).

So a PSNE exists, and the analysis progresses as in the previous examples but with the order being $\geq_S$. As an example: If the game is strictly submodular, there will not exist two PSNE which are ordered by $\geq_S$ - so if, say, the Soviet Union's stock of arms is lower in one PSNE than in another, either the US or the UK must have a lower stock of arms here too.

5 Appendix: Proof of Theorem 1

As in the one-good Cournot equilibrium case studied by Novshek (1985) and Kukushkin (1994), the argument focuses on the backward reply correspondence $B : X \to 2^S$ defined in section 2. As explained in section 2, we have (the straight-forward proof is omitted):

Lemma 3 *The joint best-reply correspondence R has a fixed point, $s^* \in R(s^*)$, if and only if there exists $(Q, s) \in X \times S$ such that $g(s) = Q$ and $s \in B(Q)$.*

Let $r : S \to S$ be an order-reversing selection from R. The graph of r is the set $\text{graph}(r) = \{(s,t) \in S \times S : s = r(t)\}$. Since R is upper semi-continuous, the closure of graph(r), denoted $\overline{\text{graph}}(r)$, is contained in the graph of R, and so the sectioning of $\overline{\text{graph}}(r)$, $s \mapsto \{t \in S : (s,t) \in \overline{\text{graph}}(r)\} \in 2^S$ is an upper hemi-continuous correspondence contained in R. In the following we replace R with this sectioning and note that this does not upset lemma (3)'s "if" part when B is defined on the subcorrespondence rather than R. The next result should be compared with Condition III in Smithson (1971). In order to get a transfinite induction machinery up and running, some condition of this type is required and conveniently it turns out that upper hemi-continuity suffices for our purposes.

Lemma 4 *Consider the backwards reply map $B : X \to 2^S \cup \emptyset$. Let $\mathcal{C}$ be a maximal chain in X and $f : \mathcal{C} \to S$ an order-reversing selection from the restriction of B to $\mathcal{C}$ (if such a selection exists). Letting c_0 denote the infimum of $\mathcal{C}$, there exists $y_0 \in B(c_0)$ such that $y_0 \geq_S f(Q)$ for all $Q \in \mathcal{C}$.*

Proof: *Let $(c_n)_{n\in\mathcal{N}}$ be a sequence of elements in $\mathcal{C}$ such that $c_n \geq c_{n+1}$ for all n and $\lim_{n\to\infty} c_n = c_0$. Since $\mathcal{C}$ is maximal, it will be a closed subset of X (Ward (1954), lemma 4). Since X is first countable - it is, in fact, Hausdorff - any point in X can be reached as the limit of a sequence from X (and an infimum evidently by a decreasing sequence). Since f is order-reversing, $f(c_n) \leq f(c_{n+1})$ for all n, which implies that for all $m \in \mathbb{N}$: $y_0 := \lim_{n\to\infty} f(c_n) \geq_S f(c_m)$ because the order is closed. y_0 is a well-defined element of X because X is compact and so chain-complete. Since $y_n = f(c_n) \in B(c_n)$ for all n, $y_0 \in B(c_0)$ because B has a closed graph.* □

For the following we need some more definitions. A subset $D \subseteq X$ of the poset X, is *directed* (*filtered*) if every two-element subset of D has an upper (lower) bound in D. A poset is a *directed complete partial order* (*DCPO*) if every directed subset has a supremum, and it is a *filtered complete partial order* (FCPO) if every filtered subset has an infimum. A map $f : X \to Y$ between two DCPOs is *Scott-continuous* if it is order-preserving and preserves directed sups, the latter meaning that $f(\sup D) = \sup f(D)$ for any directed subset $D \subseteq X$. For an order-reversing function f on a DCPO, we say that f *reverses directed sups* if for any directed subset $D \subseteq X$ it holds that,[38]

$$f(\sup D) = \inf f(D) \tag{15}$$

By assumption (iii) of the theorem, there exists an upper bound $Q^* \in X$ such that $g(s) \leq Q^*$ whenever $s \in S$. We may assume that $B(Q^*) \neq \emptyset$ without loss of generality for if this is not the case, each best-reply function can be extended along the lines of Kukushkin (1994).[39] Next define a collection $\mathcal{P}$ of subsets P of X by:

(P1) $Q^* \in P$

[38] Note that (15) is well-defined if X is a DCPO and Y is a FCPO since $f(D)$ will be filtered.

[39] See Kukushkin (1994), p.24, immediately following Proposition 1 where the extension of each ϕ_i exactly plays the role of making the backward reply correspondence well-defined at Q^* (Q^* corresponds to m in Kukuskin's notation).

(P2) If $Q \in P$, $Z \in X$, and $Q \leq Z \leq Q^*$, then $Z \in P$.

(P3) If $Q \in P$ then $M(Q) = \{s \in B(Q) : g(s) \leq Q\} \neq \emptyset$.

(P4) Let $M : P \to 2^S$ be as defined under (P3). There exists an order-reversing selection from M, $f = (f_1, \ldots, f_I) : P \to S$ which is maximal in the sense that for all $Q \in P$, $s \in M(Q)$ and $s \geq_S f(Q)$ implies $s = f(Q)$.

In the following we shall refer to a selection f which is maximal in the sense of (P4), as a *maximal order-reversing selection*. This should not be confused with a *greatest selection*, a term we reserve for the case $f(Q) = \sup M(Q)$ which obviously requires that each $M(Q)$ has a supremum (and this is rarely the case). It is an immediate consequence of (P1) and (P2) that any $P \in \mathcal{P}$ is a DCPO thus when, in the following, we speak of the supremum of a directed subset of P; this is a well-defined statement.

Lemma 5 *Order $\mathcal{P}$ by inclusion, i.e., consider the poset $(\mathcal{P}, \subseteq)$. There then exists a subset $P^* \in \mathcal{P}$ which is maximal* ($P \in \mathcal{P}$ and $P^* \subseteq P \Rightarrow P = P^*$).

Proof: *$\mathcal{P}$ is non-empty since $\{Q^*\} \in \mathcal{P}$. If $(P_m)_{m \in \mathbb{N}}$ is a chain in $(\mathcal{P}, \subseteq)$, then $\cup_{m \in \mathbb{N}} P_m$ is evidently an upper bound for this chain which satisfies* (P1)-(P3). *As for* (P4), *we may order the chain $(P_m)_{m \in \mathbb{N}}$ such that if f is a maximal order-reversing selection over P_m, then its restriction to P_n will be maximal order-reversing for all $n < m$. Thus we may pick a sequence $(f_m)_{m \in \mathbb{N}}$ of order-reversing functions $f_m : P_m \to S$, $P_m \subseteq P_{m+1}$, such that $f_m(x) = f_n(x)$ for all $x \in P_n$, $m > n$. This sequence of functions defines a function $f^* : \cup_{m \in \mathbb{N}} P_m \to S$: If $x \in \cup_{m \in \mathbb{N}} P_m$ then $x \in P_q$ for some $q \in \mathbb{N}$, and we then (uniquely) assign $f^*(x) := f_q(x)$. Since f^* clearly is maximal order-reversing,* (P4) *has been verified for the upper bound of the chain. Since, by Zorn's lemma, there exists a maximal element in any non-empty poset whose chains have upper bounds, $\mathcal{P}$ has a maximal element P^*.* □

The *set* of maximal subsets in the sense of the previous lemma is denoted $\mathcal{P}^*$. Note that there may be more than one such maximal subset and that for each $P^* \in \mathcal{P}^*$ there is associated a maximal order-reversing selection $f : P^* \to S$.

Lemma 6 *Let P^* be a maximal subset of $\mathcal{P}$, i.e., $P^* \in \mathcal{P}^*$, and $\mathcal{C} \subseteq P^*$ a maximal chain. Then the infimum of $\mathcal{C}$ exists and lies in $\mathcal{C}$.*

Proof: *Since every non-empty chain in X has an infimum, $\mathcal{C}$ (which is a maximal chain in X, in particular it is non-empty), has an infimum $c_0 \in X$. We wish to show that $c_0 \in \mathcal{C} \subseteq P^*$. By definition, $c_0 \leq c$, for all $c \in \mathcal{C}$, and c_0 is an upper bound for the elements with this property.* (P1)-(P2) *are trivially satisfied for $\mathcal{C} \cup \{c_0\}$. By* (P4), *there exists a maximal order-reversing selection $f : P^* \to S$. Clearly the restriction of f to $\mathcal{C}$, call it $f_{|\mathcal{C}}$, is maximal order-reversing on $\mathcal{C}$. We now apply lemma 4 to $f_{|\mathcal{C}} : \mathcal{C} \to S$. This allows us to conclude that $f_{|\mathcal{C}}$ has an order-reversing extension to $\mathcal{C} \cup \{c_0\}$: There exists $y \in B(c_0)$ such that $y \geq_S f_{|\mathcal{C}}(Q)$ for all $Q \in \mathcal{C}$. Studying the details of the proof of lemma 4, we notice that $y := \lim_{n \to \infty} f(c_n)$ where $(c_n)_n$ is a decreasing sequence from $\mathcal{C}$. From this follows that* (P3) *will be satisfied for $\{c_0\}$. Since the set $M(c_0) \cap \{z \in S : z \geq_S y\}$ is the intersection of a compact and a closed sets it is compact, and since the order is closed, we are then able to apply Zorn's lemma once again to conclude that it contains a maximal element, $y^* \in H(c_0) \cap \{z \in S : z \geq_S y\}$ (for a general statement and a detailed proof of this type of application of Zorn's lemma, see theorem 1 in Ward (1954)). Extending $f_{|\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{C} \cup \{c_0\}$ is then done simply by defining $f_{|\mathcal{C}}(c_0) = y^*$. The outcome of this somewhat lengthy argument, is that the extension thus constructed will be maximal order-reversing in the sense of* (P4). *But then $c_0 \in \mathcal{C}$ for if this were not the case, we would have reached a contradiction with the first part of the lemma (P^* would not be a maximal subset of X with the properties* (P1)-(P4)*).* □

By the *boundary* of P, written ∂P, we mean the subset of minimal elements of P. It is clear that any minimal element is the infimum of a maximal chain so by lemma 6, $\partial P \subset P$ when P is maximal in $(\mathcal{P}, \subseteq)$. The *interior* of P, P°, is the set $P \backslash \partial P$ and by an *interior net* we mean a net which is a subset of P°. The next result shows that maximal order-reversing selections always reverse directed sups in the sense defined above.

Lemma 7 *Let $P \in \mathcal{P}$ and $f : P \to S$ a maximal order-reversing selection as defined under* (P4). *Then $f(\sup D) = \inf f(D)$ for any directed subset $D \subseteq P$, i.e., f reverses directed sups on P.*

Proof: *Since f is order-reversing, $f(d) \geq f(\sup D)$ for all $d \in D$, i.e., $f(\sup D)$ is a lower bound of $f(D)$. Certainly then* $\inf f(D) \geq f(\sup D)$. *Since M is u.h.c.,* $\inf f(D) \in M(\sup D)$. *But then* $\inf f(D) = f(\sup D)$ *because if not* $\inf f(D) > f(\sup D)$ *contradicting the maximality of $f(\sup D)$ in $M(\sup D)$.* □

The previous proof applies more generally. If F is any u.h.c. correspondence and f is a maximal order-reversing selection (an order-reversing selection of maximal elements) from F; then this selection reverses directed sups. We will be using this observation in a moment. Among the direct consequences of lemma 7 is that f is "continuous from below" on P: If (Q^n) is an increasing sequence (or, more generally, a net); then $\lim_n f(Q^n) = \inf_n f(Q^n) = f(Q)$ when Q is the supremum of (Q^n). Additionally, it is straight-forward to adapt proposition II-2.1. in Gierz et al. (2003) in order to conclude that for any interior net (Q_n) in P,

$$f(\liminf Q_n) \geq \limsup f(Q_n) \tag{16}$$

Note (16)'s close proximity with the standard definition of upper semi-continuity (which parallels Scott-continuity's relationship with lower semi-continuity as explained in Gierz et al. (2003), chapter II). Recall that we have assumed that the aggregator $g : S \to X$ is an order-preserving *and* continuous function. Consequently,

$$g(f(\liminf Q_n)) \geq g(\inf \sup f(Q_n)) = \inf_n g(\sup_{m \geq n} f(Q_m)) \geq \limsup g(f(Q_n)) \tag{17}$$

which is to say that the composition $g \circ f : P \to X$ (also) reverses directed sups.

To simplify notation, define $R(h(Q,s)) = (R_i(h_i(Q,s_i))_{i \in \mathcal{I}}$.

Lemma 8 *Let $P \in \mathcal{P}$ and $f : P \to S$ be a maximal order-reversing selection. Then for all $Q \in P$ and $i \in \mathcal{I}$:*

$$f_i(Q) = \sup R_i(h_i(Q, f_i(Q))) \tag{18}$$

where the supremum is taken on $(S_i, \geq_i)$.

Proof: *Imagine not, i.e., that there exists some $z >_i 0$ such that $f_i(Q) + z >_i f_i(Q)$ and $f_i(Q) + z \in R_i(h_i(Q, f_i(Q)))$. But then $f_i(Q) + z \in R_i(h_i(Q + \delta, f_i(Q) + z))$ where $\delta > 0$ is such that $h_i(Q, f_i(Q)) = h_i(Q + \delta, f_i(Q) + z)$ (such a δ exists because h_i is continuous). Note that $f_i(Q) + z \in B_i(Q + \delta)$ and $F_i(x, f_i(Q)) = Q$, $F_i(x, f_i(Q) + z) = Q + \delta$ where $x = h_i(Q, f_i(Q))$. Clearly, $x \geq g(f_{-i}(Q), \theta_i)$, for if $x^m < g^m(f_{-i}(Q), \theta_i)$, some m, then $g^m(f_i(Q)) = F_i^m(g^m(f_{-i}(Q), \theta_i), f_i^{\mathcal{M}(m)}(Q)) > F_i^m(x^m, f_i^{\mathcal{M}(m)}(Q)) = Q^m$. But then $Q + \delta = F_i(x, f_i(Q) + z) \geq F_i(g(f_{-i}(Q), \theta_i), f_i(Q) + z) \geq F_i(g(f_{-i}(Q + \delta), \theta_i), f_i(Q) + z) = g(f_{-i}(Q + \delta), f_i(Q) + z)$. A contradiction.* □

The previous lemma implies that we from now on can focus on the greatest selection from R, $r(z) = \sup_{z \in S} R(z)$ (we risk confusion by denoting this greatest selection by r although it may of course differ from the original order-reversing selection from the original R). Letting $r(h(Q)) = (r_i(h_i(Q, s_i))_{i \in \mathcal{I}}$, it follows in particular that the maximal order-reversing selection f, will be a selection from $b(Q) = \{s \in S : s = r(h(Q,s))\} \subseteq B(Q) = \{s \in S : s \in R(h(Q,s))\}$.

Lemma 9 *If for $Q^1 \leq Q^2$ there exists $s^1 \in b(Q^1)$ and $s^2 \in b(Q^2)$ with $s^2 \leq_S s^1$, then for all $Q \in [Q^1, Q^2] \cap X$ the set $b(Q) \cap [s^2, s^1]$ is a non-empty complete lattice. Consequently if $g(s^1) \leq Q^1$ then $g(\sup\{b(Q) \cap [s^2, s^1]\}) \leq Q$ for all $Q \in [Q^1, Q^2]$.*

Proof: *Since h is order-reversing in s and order-preserving in Q, $r^Q(s) \equiv r(h(Q,s))$ is order-reversing in Q and order-preserving in s. It follows that for $Q \in [Q^1, Q^2]$: $r_i(h_i(Q, s_i^1)) \leq_i s_i^1$ and $r_i(h_i(Q, s_i^2)) \geq_i s_i^2$ for all i $\Rightarrow r^Q : [s_2, s_1] \to [s_2, s_1]$. Since $[s_2, s_1] = \{s \in S : s_2 \leq_S s \leq_S s_1\}$ is a complete lattice, the lemma's conclusion now follows from Tarski's fixed point theorem after observing that $b(Q) \cap [s_2, s_1] = \{s \in [s_2, s_1] : s = r(h(Q,s))\}$. The mentioned consequence is obvious.* □

In the remainder we fix a maximal subset P^* in accordance with lemma 5. By (P4), there is defined upon P^* a maximal order-reversing function $f : P^* \to S$ ($f = (f_i)_{i\in\mathcal{I}}$) which to each $Q \in P^*$ associates a joint backward best-reply $f(Q) \in H(Q) \subseteq S$. If $g(f(c_0)) = c_0$ for some $c_0 \in P^*$, this implies the conclusion of the theorem by lemma 3. Thus we must study the situation where $g(f(c_0)) < c_0$ and shall do so from now on.

The next lemma is a direct adaption of the main argument of Novshek (1985) and Kukushkin (1994).

Lemma 10 *Pick $c_0 \in \partial P^*$. For all $i \in \mathcal{I}$ and $\epsilon_i \in \Omega_i \equiv \{\epsilon >_i 0 : g(f_{-i}(c_0), f_i(c_0) + \epsilon) \leq c_0\}$, it holds that:*

$$r_i(h_i(c_0, f_i(c_0) + \epsilon_i)) <_i \epsilon_i + f_i(c_0) \tag{19}$$

Proof: *By contradiction. Assume that $\geq$ holds for some $i \in \mathcal{I}$ and some $\epsilon \in \Omega_i$. It is clear that equality cannot hold since this would mean that $\epsilon + f_i(c_0) \in B_i(c_0)$. Define $w_i = h_i(c_0, f_i(c_0)+\epsilon)$ (so $c_0 = F_i(w_i, f_i(c_0)+\epsilon)$) and $c_i = F_i(w_i, r_i(w_i)) = F_i(h_i(c_0, f_i(c_0) + \epsilon), r_i(w_i)) \geq F_i(h_i(c_0, f_i(c_0) + \epsilon), f_i(c_0) + \epsilon) = c_0$. Here the last equality holds because $h_i(y, z) = x \Leftrightarrow F_i(x, z) = y$. We now show that $((f_j(c_i))_{j\neq i}, r_i(w_i)) \in M(c_i)$. Firstly, $r_i(w_i) \in B_i(c_i)$ since $r_i(w_i) = r_i(h_i(c_0, f_i(c_0) + \epsilon)) = r_i(h_i(c_i, r_i(w_i)))$.*[40] *Secondly, $g(((f_j(c_i))_{j\neq i}, r_i(w_i)) \leq c_i$, because $g(f_{-i}(c_0), f_i(c_0) + \epsilon) \leq c_0 \Leftrightarrow F_i(g(f_{-i}(c_0), \theta_i), f_i(c_0) + \epsilon) \leq F_i(w_i, f_i(c_0) + \epsilon) = F_i(h_i(c_i, r_i(w_i)), f_i(c_0) + \epsilon)$ implying that $g(f_{-i}(c_0), \theta_i) \leq h_i(c_i, r_i(w_i))$ because F_i is strictly increasing in its first coordinate. But (again because F_i is strictly increasing in its first coordinate), this implies the conclusion: Let $h_i(c_i, r_i(w_i)) = x$ then $c_i = F_i(x, r_i(w_i)) \geq F_i(g(f_{-i}(c_0), \theta_i), r_i(w_i)) = g(f_{-i}(c_0), r_i(w_i))$. But we also have that $g(f_{-i}(c_0), r_i(w_i)) > g(f(c_i))$, because $r_i(w_i) >_i f_i(c_0) \geq_i f_i(c_i)$. Here the second inequality is true because $c_i \geq c_0$ and f_i is order-reversing. This contradicts the fact that $f : [c_0, Q^*] \to S$ is maximal order-reversing. Indeed, $f^* : [c_i, Q^*] \to S$, where $f^*(c_i) = ((f_j(c_i))_{j\neq i}, r_i(w_i))$ and $f^*(c) = f(c)$ otherwise, is order-reversing and strictly larger than f at c_i.* □

Lemma 11 *Let $c \in P^*$ and assume that $g(f(c)) \ll c$. Then for every $i \in \mathcal{I}$ and every $n \in \{1, \ldots, N\}$ there exists $\delta_i \geq 0$ with $g(f_{-i}(c), f_i(c) + \delta_i) \leq c$, such that $r_i(h_i(c, f_i(c) + \delta_i)) \leq \delta_i + f_i(c)$ with strict inequality in coordinate n.*

Proof: *The result is automatic if $N = 1$ and straight-forward for $N = 2$. To save space, we shall prove the claim for $N = 3$ and leave the extension to $N > 3$ (which gets very lenghty but adds nothing new) to the interested reader. Define $\phi_i(\epsilon) = r_i(h_i(c, f_i(c)+\epsilon) - f_i(c)$. For every $\epsilon = (\epsilon^1, \epsilon^2, \epsilon^3) \in \Omega_i$ we have by the previous lemma some n_1 such that $\phi_i^{n_1}(\epsilon) < \epsilon^{n_1}$. Use Tarski's fixed point theorem to get $\epsilon^{n_1} > \epsilon_1^{n_1} \geq \phi_i^{n_1}(0^{n_1}, \epsilon^{-n_1})$ such that $\phi_i^{n_1}(\epsilon_1^{n_1}, \epsilon^{-n_1}) = \epsilon_1^{n_1}$. Since $(\epsilon_1^{n_1}, \epsilon^{-n_1}) \in \Omega_i$, we can apply lemma 10 once more, yielding $n_2 \neq n_1$ such that $\phi_i^{n_2}(\epsilon_1^{n_1}, \epsilon^{-n_1}) < \epsilon^{n_2}$. Fixing ϵ^{n_3} observe that*

$$\phi_i^{n_1,n_2}(\cdot, \cdot, \epsilon^{n_3}) : [(0, 0), (\epsilon_1^{n_1}, \epsilon^{n_2})] \to [(0, 0), (\epsilon_1^{n_1}, \epsilon^{n_2})]$$

Thus by Tarski's fixed point theorem, there is a maximal fixed point $(\epsilon_2^{n_1}, \epsilon_2^{n_2})$ such that $\phi_i^{n_1,n_2}(\epsilon_2^{n_1}, \epsilon_2^{n_2}, \epsilon^{n_3}) = (\epsilon_2^{n_1}, \epsilon_2^{n_2})$. Lemma 10 implies that $\phi_i^{n_3}(\epsilon_2^{n_1}, \epsilon_2^{n_2}, \epsilon^{n_3}) < \epsilon^{n_3}$. This immediately implies the conclusion of the lemma for n_3. Additionally, if $\epsilon_2^{n_1} > 0$ (resp., $\epsilon_2^{n_2} > 0$) then we can fix this $\epsilon_2^{n_1}$ and apply Tarski's fixed point theorem on the remaining two coordinates, the set $\{-n_1\}$, and so get the conclusion of the lemma in n_1 (resp., n_2). So the remaining situations to be considered is when $\epsilon_2^{n_1} = 0$ and/or $\epsilon_2^{n_2} = 0$. Imagine that $\epsilon_2^{n_1} = 0$ and consider $\phi_i^{n_2}(0, \epsilon_2^{n_2}, \epsilon^{n_3}) = \epsilon_2^{n_2}$. Now raise $\epsilon_2^{n_2}$ slightly to $\epsilon_3^{n_2} > \epsilon_2^{n_2}$. By left-continuity of r (r is a greatest selection from an order-reversing u.h.c. correspondence, cf. the remarks following lemma 7), we still have: $\phi_i^{n_3}(0, \epsilon_3^{n_2}, \epsilon^{n_3}) < \epsilon^{n_3}$. We must have $\phi_i^{n_2}(0, \epsilon_3^{n_2}, \epsilon^{n_3}) < \epsilon_3^{n_2}$ since if not, we get the following contradiction: Assuming $\phi_i^{n_2}(0, \epsilon_3^{n_2}, \epsilon^{n_3}) \geq \epsilon_3^{n_2}$,

$$\phi_i^{n_1,n_2}(\cdot, \cdot, \epsilon^{n_3}) : [(0, \epsilon_3^{n_2}), (\epsilon_1^{n_1}, \epsilon^{n_2})] \to [(0, \epsilon_3^{n_2}), (\epsilon_1^{n_1}, \epsilon^{n_2})] \ ,$$

so $(0, \epsilon_2^{n_2})$ cannot be the maximal fixed point in the second step above. By left-continuity, we can now raise $\epsilon_2^{n_1}$ slightly from 0 to $\epsilon_3^{n_1} > 0$ and will still have strict inequalities in n_2 and n_3. But then we can extend, since we simply iterate on n_2 and n_3 holding $\epsilon_3^{n_1} > 0$ fixed which produces $(\epsilon_3^{n_2}, \epsilon_3^{n_3})$ such that $\phi_i^{n_1}(\epsilon_3^{n_1}, \epsilon_3^{n_2}, \epsilon_3^{n_3}) < \epsilon_3^{n_1}$ while equality holds for n_2 and n_3. Next imagine that $\epsilon_2^{n_2} = 0$ and consider $\phi_i^{n_1}(\epsilon_2^{n_1}, 0, \epsilon^{n_3}) = \epsilon_2^{n_1}$. Now raise $\epsilon_2^{n_1}$

[40] Here use: $h_i(F_i(w_i, r_i(w_i)), r_i(w_i)) = w_i$, which is true because $h_i(Q, r_i(w_i)) = w_i \Leftrightarrow h_i(Q, r_i(w_i)) = w_i$.

slightly to $\epsilon_3^{n_1} > \epsilon_2^{n_1}$*. By left-continuity, we still have:* $\phi_i^{n_3}(\epsilon_3^{n_1}, 0, \epsilon^{n_3}) < \epsilon^{n_3}$*. If we have* $\phi_i^{n_1}(\epsilon_3^{n_1}, 0, \epsilon^{n_3}) < \epsilon_3^{n_1}$ *we can proceed as above; so assume not,* $\phi_i^{n_1}(\epsilon_3^{n_1}, 0, \epsilon^{n_3}) \geq \epsilon_3^{n_1}$*. It holds that,*

$$\phi_i^{n_1,n_2}(\cdot, \cdot, \epsilon^{n_3}) : [(\epsilon_3^{n_1}, 0), (\epsilon_1^{n_1}, \epsilon^{n_2})] \to [(\epsilon_3^{n_1}, 0), (\epsilon_1^{n_1}, \epsilon^{n_2})]$$

Again this leads to a contradiction - but this time provided that $\epsilon_3^{n_1} \leq \epsilon_1^{n_1}$ *which can be ensured iff* $\epsilon_2^{n_1} < \epsilon_1^{n_1}$ *(we always have weak inequality). But* $\epsilon_2^{n_1} = \epsilon_1^{n_1}$ *implies that* $\phi_i^{n_1}(\epsilon_1^{n_1}, 0, \epsilon^{n_3}) = \phi_i^{n_1}(\epsilon_1^{n_1}, \epsilon^{n_2}, \epsilon^{n_3})$ *(*$\phi_i^{n_1}$ *is constant in the second term). If so, we can instead raise* $\epsilon_2^{n_2} = 0$ *above zero, and will have equality in the coordinate* n_1 *while as before inequality holds in* n_3*. Consequently, we can iterate on* $\{-n_2\}$ *and so extend in coordinate* n_2*.* □

The first important conclusion one is able to draw from lemma 11 is that when $c \in \partial P^*$ it *cannot* be the case that $g(f(c)) \ll c$. Consequently the proof is complete already here in the one-dimensional case. Specifically we will be needing the following way of stating this:[41]

Lemma 12 *Define upon* $P^* \in \mathcal{P}^*$*, a function* $F : P^* \to \mathbb{R}_+^M$ *by the assignment,*

$$F(Q) = Q - g(f(Q)) \tag{20}$$

where f *is the maximal order-reversing selection associated with* P^**. Then* $F(\partial P^*) \subseteq \partial \mathbb{R}_+^M$*, i.e.,* F *maps the boundary of* P^* *to the boundary of* $\mathbb{R}_+^M$*.*

Proof: *By contradiction. Use lemma* 10 *to pick* $\epsilon_i \in \Omega_i$ *for every* i *and define* $\delta_i = f_i(c_0) + \epsilon_i - r_i(h_i(c_0, f_i(c_0) + \epsilon_i)) >_i 0$*. Since* $r_i(h_i(c_0, \cdot))$ *is order-preserving,* $\epsilon_i - \delta_i >_i 0$*. By lemma 11, we can find* $(\epsilon_i)_{i \in \mathcal{I}}$ *and a (fixed)* $n \in \{1, \ldots, N\}$ *such that* $\delta_i^n > 0$ *for all* i*. Next, for every* i *pick* $\tilde{\delta}_i > 0$ *such that* $h_i(c_0, f_i(c_0) + \epsilon_i) = h_i(c_0 - \tilde{\delta}_i, f_i(c_0) + \epsilon_i - \delta_i)$ *(possible by continuity). By definition of* B_i *then* $f_i(c_0) + \epsilon_i - \delta_i \in B_i(c_0 - \tilde{\delta}_i)$*. It is clear that in this construction,* $\delta_i^{\mathcal{M}(m)} > 0 \Rightarrow \tilde{\delta}_i^m > 0$*, and so* $\tilde{\delta}_i^n > 0$ *for all* i*. Now lemma 9 comes into play. This lemma together with the previous observation imply that for each* i *and every* $\rho \in [0, \tilde{\delta}_i]$*, there exists* $s_i \in B_i(c - \rho)$ *with* $f_i(c) \leq s_i \leq f_i(c_0) + \epsilon_i - \delta_i$*. Taking* $\hat{\rho} = (\inf\{\tilde{\delta}_1^n, \ldots, \tilde{\delta}_I^n\}, 0^{-n})$*, then yields* $s = (s_1, \ldots, s_I)$ *s.t.* $s \in B(c - \hat{\rho})$ *and* $f_i(c) \leq s_i \leq f_i(c_0) + \epsilon_i - \delta_i$ *for all* i*. It follows that* $g(s_1, \ldots, s_I) \leq g((f_i(c_0) + \epsilon_i - \delta_i)_{i \in \mathcal{I}})$*. Picking* $(\epsilon_1, \ldots, \epsilon_I)$ *such that* $g((f_i(c_0) + \epsilon_i)_{i \in \mathcal{I}}) \leq c_0$*, this implies that the maximal chain* $\mathcal{C} \subseteq P^*$ *whose infimum is* $c \in \partial P^*$ *can be extended to* $[c - \hat{\rho}, c] \cup \mathcal{C}$*. This contradicts the maximality of* P^**.*[42] □

Note that $g(f(\liminf Q_n)) \geq \limsup g(f(Q_n)) \Leftrightarrow -g(f(\liminf Q_n)) \leq \liminf -g(f(Q_n))$, and so, because $\liminf a_n + \liminf b_n \leq \liminf[a_n + b_n]$ (which always holds when the terms involved are well-defined and the order is closed):

$$F(\liminf Q^n) \leq \liminf F(Q^n) \tag{21}$$

By (17), (21) consequently holds for any interior sequence $(Q_n)_n$ in P^*. The next lemma extends this statement to all of P^*. We shall omit the proof (it is available from the author upon request); because it forces us to introduce a considerable amount of extra mathematics (Painleve-Kuratowski limit, nets, Cantor diagonalization).

Lemma 13 (21) *holds for any sequence* $(Q^n)_n$ *in* P^**.*

For a sequence $(x^n)_n$, say that x is an $\mathcal{L}$-limit, written $x \equiv_{\mathcal{L}} \lim_n x_n$ provided that $\liminf x^n \geq_X x$. From this define a family of subsets:[43]

$$\mathcal{O}(X) = \{U \subseteq X : \text{ whenever } x \equiv_{\mathcal{L}} \lim_n x_n \text{ and } x \in U, \text{ then eventually } x^n \in U\} \tag{22}$$

The family $\mathcal{O}(X)$ is a topology, more precisely it is the topology generated by the convergence classes consisting of those pairs $((x^n)_n, x)$ for which $x \equiv_{\mathcal{L}} \lim_n x_n$ (see Kelley (1955), chapter 2, especially theorem 9).

[41] Note that the positive cone of X is here taken to be the positive cone of the usual order. This is possible by the remarks preceding lemma 10.

[42] Note that this extension actually requires us to repeat the above argument for all points $d \in [c - \hat{\rho}, c]$. Clearly this poses no problem since lemma 9 applies to the entire interval $[c - \hat{\rho}, c]$. Also note that it is indeed possible to pick $(\epsilon_1, \ldots, \epsilon_I)$ "sufficiently small" as described (this is easily seen from the proof of lemma 11).

[43] A sequence $(x^n)_n$ is *eventually* in a set U if there is some $k \in \mathbb{N}$ such that $n \geq k$ implies $x^n \in U$.

The topology $\mathcal{O}(X)$ is essentially the Scott-topology (Gierz et al. (2003)), although "tweaked" so as to work well on the boundary ∂P^* also ($\mathcal{O}(X)$ and the Scott-topology agree on the interior of P^*). Notice the very close relationship with lower semi-continuity (with lower semi-continuity one would replace $\liminf Q^n$ with a limit taken with respect to the usual topology; something we shall encounter shortly). If a sequence $(x^n)_n$ converges to x in the usual topology, it is clearly the case that x is an $\mathcal{L}$-limit of $(x^n)_n$ (which is to say that $\mathcal{O}(X)$ is coarser than the usual topology). On the other hand $(X, \mathcal{O}(X))$, while being T_1 is not Hausdorff since $\mathcal{L}$-limits are far from being unique. Comparing with the statement in lemma 13, it is seen that $F : P^* \to \mathbb{R}^M_+$ is *continuous* with respect to the topologies $\mathcal{O}(P^*)$ and $\mathcal{O}(\mathbb{R}^M_+)$ (subsets are as always endowed with the induced topology). We shall say simply that F is continuous from now on.

Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$ (so $\mathbb{S}^1$ is the unit circle, $\mathbb{S}^2$ the unit-sphere in $\mathbb{R}^3$, and so forth) endowed with the usual topology. Two maps $g_1, g_2 : \mathbb{S}^n \to \partial P^*$ are homotopic if there exists a continuous map $H : \mathbb{S}^n \times [0,1] \to \partial P^*$ (the homotopy) such that $H(z,0) = g_1(z)$ and $H(z,1) = g_2(z)$ for all $z \in \mathbb{S}^n$. We write in this case $g_1 \cong g_2$. For a fixed base point $Q_0 \in \partial P^*$, a map $g : \mathbb{S}^n \to \partial P^*$ is nullhomotopic if g and the constant map $1 : \mathbb{S}^n \to \{Q_0\}$ are homotopic. The n'th homotopy group is denoted $\pi_n(\partial P^*)$ and this is trivial if it consists only of the single equivalence class [1]. A set is n-connected if $\pi_m(\partial P^*)$ is trivial for all $m \leq n$ (here, following the standard conventions, -1-connected means non-empty, and 0-connected means path-connected). By our assumptions, $X = g(S)$ is a convex, compact set, and it is easily seen that ∂P^* is therefore contractible (in fact, it is an absolute retract of X). This then implies that these sets are n-connected for all n.

Let $\sigma^m = \{x \in F(\partial P^*) : x_m = 0\}$. From now on we refer to these M subsets of $\mathbb{R}^M_+$ as the *components*. By lemma 12, $F : \partial P^* \to \cup_{m=1}^M \sigma^m$. Let $Q^* = (Q^{*,1}, \ldots, Q^{*,M}) \in X$. Assume w.l.g. that ∂P^* has full (covering) dimension, *i.e.*, dimension equal to $M-1$ being the lower boundary of an M-dimensional Euclidean manifold (if the dimension is lower, simply use the following argument in the lower-dimensional case). Since P^* is an upper set in a product set X, there will for any $m \in \{1, \ldots, M\}$ exist elements $\hat{Q}^m = (Q^m, Q^{*,-m}) \in \partial P^*$.

Lemma 14 *(i) For all m, $F(\hat{Q}^m) \in \sigma^m$. (ii) For any two $m_1 \neq m_2$, there exists a continuous map $r : [0,1] \to \sigma^{m_1} \cup \sigma^{m_2}$ with $r(0) = F(\hat{Q}^{m_1})$ and $r(1) = F(\hat{Q}^{m_2})$.*

Proof: *(i) If $F^m(Q^m, Q^{*,-m}) < 0$ were to hold, we could use the fact that $F^j(\tilde{Q}^m, Q^{*,-m}) \leq 0$ for all $j \neq m$ and all $\tilde{Q}^m \leq Q^m$ (remember that Q^* is an upper bound !) in conjunction with the argument used in lemma 11 to obtain an extension and contradict the maximality of P^*. (ii) Given m_1 and m_2 let $A = \{Q \in \partial P^* : Q^j = Q^{*,j}$ for all $j \notin \{m_1, m_2\}\}$ which is a path-connected subset of ∂P^*. Clearly, $\hat{Q}^{m_1}, \hat{Q}^{m_2} \in A$. For $Q \in A$, $F(Q) \in \sigma^{m_1} \cup \sigma^{m_2}$ for if not the argument from (i) would apply and lead to a contradiction. The conclusion is now trivial (simply choose a path in A connecting $\hat{Q}^{m_1}$ with $\hat{Q}^{m_2}$ and let r be the composition with F).*[44] □

Pick $s_1, \ldots, s_M \in \mathbb{S}^1$ clockwise ordered and with equal distance between subsequent elements. Fix s_1 as the basepoint. Now let $g : \{s_1, \ldots, s_M\} \to \partial P^*$ be such that $g(s_m) = \hat{Q}^m$ for all m. By lemma 14, we can extend g to the whole unit simplex in such a way that the extension is continuous and $g(s) \in \sigma^m \cup \sigma^{m+1}$ whenever $s \in [s_m, s_{m+1}]_{\mathbb{S}^1}$ (here $M+1 := 1$). The next observation is the key to our proof. It says intuitively that although $F \circ g$ may well "jump" it cannot jump "between" any two different components. Because $\mathcal{O}(P^*)$ is coarser than the original topology, we have therefore that if $g : \mathbb{S}^1 \to \partial P^*$ is any continuous function (where both sets are given the original topology), then the composition $F \circ g : \mathbb{S}^1 \to \mathbb{R}^M_+$ is continuous. But the same statement is valid if ∂P^* is endowed with the topology $\mathcal{O}(\partial P^*)$ (which weakens the continuity requirement on g). It is in fact this stronger statement which is made here (and consequently the conclusion applies if g is merely lower semi-continuous):

Lemma 15 *Let $s_n \to s$ be any convergent sequence which is such that for some m, $F(g(s_n)) \in \sigma^m$ all n. Then $F(g(s)) \in \sigma^m$.*

[44]For every $m \in \{1, \ldots, M\}$ let $\partial P^*_{|Q^{*,m}}$ denote the section $\{Q \in \partial P^* : Q^m = Q^{*,m}\}$. If $Q \in \partial P^*_{|Q^{*,m}}$, it cannot be true that $F^m(Q) = 0$ while $F^j(Q) > 0$ for all $j \neq m$ (the reason is as before: If $F^m(Q) = 0$ then $F^m(\tilde{Q}) = 0$ for all $\tilde{Q} \leq Q$ and so we can extend and contradict the maximality of P^*).

Proof: *By continuity of g and F, $g(s) \leq \liminf g(s_n)$ and $\liminf F(g(s_n)) \geq F(\liminf g(s_n))$. Because F is order-preserving, $\liminf F(g(s_n)) \geq_X F(g(s))$. Since $F^m(g(s_n)) = 0$ for all n, $\liminf F(g(s_n)))^m = 0$ implying that $F^m(g(s)) = 0$.* □

As mentioned, lemma 15 says intuitively that $F \circ g$ cannot jump between any two components $\sigma^{m_1} \neq \sigma^{m_2}$, so whenever a loop moves between such components, it passes through a point in their intersection. More formally, for any continuous map $r : [0, 1] \to F(\partial P^*) \cap (\sigma^{m_1} \cup \sigma^{m_2})$ with $r(0) \in \sigma^{m_1}$, $r(1) \in \sigma^{m_2}$ there must be $t \in [0, 1]$ with $r(t) \in \sigma^{m_1} \cap \sigma^{m_2}$. No path can ever jump across the origin $0 \in \mathbb{R}^M_+$ because this is the infimum of $\cup_{m=1}^M \sigma^m$ and any jump is a jump down in the sense of (21). In the three-dimensional case, the argument is now fairly simple: The loop $F \circ g : \mathbb{S}^1 \to \cup_{m=1}^3 \sigma^m$ runs through all components. No matter how the homotopy $H : \mathbb{S}^1 \times [0, 1] \to \cup_{m=1}^3 \sigma^m$ deforms this loop to the basepoint $F \circ g(s_1)$, it will for any t still run through all σ^m's unless it has beforehand (for some $t' < t$) been drawn through 0 (and, in particular, the loop cannot change direction, *i.e.*, there are no singularities). Indeed, by lemma 15 it is not allowed to jump between different components. But since the loop does deform continuously to its basepoint $F(g(s_1))$ (where s_1 is the basepoint in $\mathbb{S}^1$); in the limit (when $t \to 1$ in the homotopy); any sequence in $\mathbb{S}^1$ converging to s_1 will induce a sequence in $F(\partial P^*)$ whose *liminf* is greater than or equal to $F(g(s_1))$. But if the loop passes through every σ^m for all $t < 1$; this is impossible unless $F(s_1) = 0$. Indeed, from every σ^m could then be picked a sequence whose *liminf* must be greater than or equal to $F(s_1)$ and by the definition of σ^m it is clear that then $F(s_1) = 0$ (each σ^m forces one coordinate to 0).

In the general case ($M > 3$), we have $\{s_1, \ldots, s_M\} \subset \mathbb{S}^{M-2}$ and an assignment $g : \{s_1, \ldots, s_M\} \to \partial P^*$ such that $g \circ F(s_m) \in \sigma^m$ for all m. For any two points s_{m_1} and s_{m_2} we can find a continuous function $r : [0, 1]^{M-2} \to \sigma^{m_1} \cup \sigma^{m_2}$. It is still true that no path can jump between components, so if we restrict r to a path, $w : [0, 1] \to \sigma^{m_1} \cup \sigma^{m_2}$ with $w(0) \in \sigma^{m_1}$ and $w(1) \in \sigma^{m_2}$, there will exist $t \in [0, 1]$ with $w(t) \in \sigma^{m_1} \cap \sigma^{m_2}$. As a consequence, we can attach the "cells" $[0, 1]^{M-2}$ at their points of intersection (which is always non-empty) and we then obtain a continuous extension of g, $g : \mathbb{S}^{M-2} \to \partial P^*$. By composition, $F \circ g : \mathbb{S}^{M-2} \to \cup_{m=1}^M \sigma^m$ is continuous. That $0 \in F(\partial P^*)$ is now proved in much the same way as above: Deforming $F \circ g(\mathbb{S}^{M-2})$ to a point via the homotopy $H(s, t)$ it will for any t still run through all components or it must be drawn through 0. Assuming this, we then look as the limit ($t \to 1$ so sequences in $\mathbb{S}^{M-2}$ converge to the basepoint s_1) and conclude that $F(g(s_1)) = 0$ by a repetition of the argument at the end of the previous paragraph.

References

Acemoglu, D. and A.E. Ozdaglar (2006): "Competition and Efficiency in Congested Markets", *Mathematics of Operations Research*, forthcoming. 3

Alos-Ferrer, C. and A.B. Ania (2002): "The Evolutionary Logic of Perfectly Competitive Behaviour", *Economic Theory*, forthcoming. 1, 3

Amir, R. (1996): "Cournot Oligopoly and the Theory of Supermodular Games", *Games and Economic Behavior* 15, 132-148. 19

Amir, R. and V.E. Lambson (2000): "On the Effects of Entry in Cournot Markets", *Review of Economic Studies* 67, 235-254. 16, 18

Bulow, J., J. Geanakoplos and P. Klemperer (1985): "Multimarket Oligopoly: Strategic Substitutes and Complements", *Journal of Political Economy* 93, 488-511. 1, 19, 20

Corchón, L. (1994): "Comparative Statics for Aggregative Games. The Strong Concavity Case", *Mathematical Social Sciences* 28, 151-165. 1, 2, 9, 12, 14, 15, 17

Curtat, L.O. (1996): "Markov Equilibria of Stochastic Games with Complementarities", *Games and Economic Behavior* 17, 177-199. 1, 2, 16

Dubey, P., A. Mas-Colell, and M. Shubik (1980): “Efficiency Properties of Strategic Market Games: An Axiomatic Approach”, *Journal of Economic Theory* 22, 339-362. 1

Dubey, P., O. Haimanko, and A. Zapechelnyuk (2002): “Strategic Complements and Substitutes, and Potential games”, *Center for Game Theory, SUNY at Stony Brook*. 1, 3, 6, 20, 21

Dufwenberg, M. and U. Gneezy (2000): “Price Competition and Market Concentration: An Experimental Study”, *International Journal of Industrial Organization* 18, 7-22. 23

Dugundji, J. and A. Granas, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer Verlag 2003. 13

Gierz, G., K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, 2003. 8, 26, 29

Granot, F. and A.F. Veinott (1985): “Substitutes, Complements, and Ripples in Network Flows”, *Mathematics of Operation Research* 10, 471-497. 2, 16

Jensen, M.K. (2003): “Unbounded Growth with Heterogenous Consumers”, *Journal of Mathematical Economics*, forthcoming.

Jensen, M.K. (2004): “Monotone Comparative Statics in Ordered Vector Spaces”, Mimeo. 6, 23

Karp,. L., I.H. Lee, and R. Mason: “A Global Game with Strategic Substitutes and Complements”, *CUDARE Working Paper nr.* 940, University of California at Berkeley, 2003. 9

Kelley, J.L., *General Topology*, Springer-Verlag, New-York, 1955. 29

Kukushkin, N.S. (1994): “A Fixed-point Theorem for Decreasing Mappings”, *Economics Letters* 46, 23-26. 1, 2, 3, 6, 7, 8, 24, 25, 27

Kukushkin, N.S. (2004): “Best Response Dynamics in Finite Games with Additive Aggregation”, *Games and Economic Behavior* 48, 94-110. 3

McManus, M. (1964): “Equilibrium, Numbers and Size in Cournot Oligopoly”, *Yorkshire Bulletin of Social Science and Economic Research* 16, 68-75. 19

Milgrom, P. and J. Roberts (1990): “Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities”, *Econometrica* 58, 1255-1277. 1, 2, 3, 9, 13, 22, 23

Milgrom, P. and C. Shannon (1994): “Monotone Comparative Statics”, *Econometrica* 62, 157-180. 2, 6, 12

Monderer, D. and L.S. Shapley (1994): “Potential Games”, *Games and Economic Behaviour* 44, 124-143. 3

Novshek, W. (1980): “Cournot Equilibrium with Free Entry”, *Review of Economic Studies* 47, 473-486. 3

Novshek, W. (1985): “On the Existence of Cournot Equilibrium”, *Review of Economic Studies* 52, 86-98. 1, 3, 6, 8, 24, 27

Possajennikov, A. (2003): “Evolutionary Foundation of Aggregative-taking Behavior”, *Economic Theory* 21, 921-928. 1, 3

Roberts, J. and H. Sonnenschein (1976): "On the Existence of Cournot Equilibrium without Concave Profit Functions", *Journal of Economic Theory* 13, 112-117. 18, 19

Roberts, J. and H. Sonnenschein (1977): "On the Foundations of the Theory of Monopolistic Competition", *Econometrica* 45, 101-113. 19

Schipper, B. (2005): "The Evolutionary Stability of Optimism, Pessimism and Complete Ignorance", *Mimeo*, University of California at Davis. 1, 3

Selten, R., *Preispolitik der Mehrproduktenunternehmung in der Statischen Theorie*, Springer Verlag, Berlin, 1970. 7

Shannon, C. (1995): "Weak and Strong Monotone Comparative Statics", *Economic Theory* 5, 209-227. 4

Smith, H.L. (1988): "Systems of Ordinary Differential Equations which Generate an Order Preserving Flow. A Survey of Results", *SIAM Review* 30, 87-113. 23

Smithson, R.E. (1971): "Fixed Points of Order Preserving Multifunctions", *Proceedings of the American Mathematical Society* 28, 304-310. 7, 24

Topkis, D.M. (1978): "Minimizing a Submodular Function on a Lattice", *Operations Research*, 26, 305-321. 1, 13, 20, 21

Topkis, D.M. (1979): "Equilibrium Points in Nonzero-sum n-person Submodular Games", *SIAM Journal of Control and Optimization* 17, 773-787. 1, 22

Topkis, D.M., *Supermodularity and Complementarity*, Princeton University Press, New Jersey, 1998. 2, 19

Villas-Boas, J.M. (1997): "Comparative Statics of Fixed Points", *Journal of Economic Theory* 73, 183-198. 1, 11

Vives, X. (1990): "Nash Equilibrium with Strategic Complementarities", *Journal of Mathematical Economics* 19, 305-321. 1, 18, 19

Vives, X., *Oligopoly Pricing: Old Ideas and New Tools*, MIT Press, Cambridge, Massachusetts, 2000. 1, 11, 12, 15, 16, 22

Ward, L.E. (1954): "Partially Ordered Topological Spaces", *Proceedings of the American Mathematical Society* 5, 144-161. 24, 25

Zhou, L. (1994): "The Set of Nash Equilibria of a Supermodular Game is a Complete Lattice", *Games and Economic Behaviour* 7, 295-300. 1, 9