Stability of Pure Strategy Nash Equilibrium in Best-reply Potential Games

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Abstract

A game with single-valued best-replies and a unique PSNE is a best-reply potential game if and only if the sequential best-reply dynamics converge (Cournot stability). If the PSNE is not unique, convergence to the set of equilibria is still ensured. It follows, for example, that (a) any strictly quasi-concave supermodular game with a unique PSNE is a best-reply potential game, (b) any strictly quasi-concave and dominance solvable game is a best-reply potential game, and (c) any strictly quasi-concave aggregative game with strategic substitutes is Cournot stable.

Keywords: Best-reply potential game, sequential best-reply dynamics, computation of pure strategy Nash equilibrium, Cournot dynamics, Dominance Solvability.

JEL-classification: C72, C62, D43.

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1 Introduction

Best-reply potential games were introduced by Voorneveld (2000) as a natural generalization of potential games (Monderer and Shapley (1996)). A game is a best-reply potential game if there exists a real-valued function (the potential), such that each player's best-reply set equals the set of maximizers of the potential (given opponents' strategies). The generalization has proved very useful, because many potential game arguments depend only on the fact that best-reply correspondences agree with those of a coordination game (Morris and Ui (2004)). The study of sequential best-reply dynamics dates back to Cournot (1838). At sequential best-reply paths, players move in turn and always to a best-reply. A large number of papers have studied convergence to a pure strategy Nash equilibrium (PSNE) of such dynamics (see Vives (2000), Section 2.6. for an overview). A result due to Voorneveld (2000) establishes that in *finite* games, all sequential best-reply paths terminate at a PSNE if and only if the game admits a best-reply potential. The present study's purpose is to establish a similar result which applies to infinite games. The result reads as follows:

A game with single-valued continuous best-reply functions and a unique PSNE is a best-reply potential game *if and only if* all sequential best-reply paths converge to the PSNE.

A PSNE which is globally stable under the game's best-reply dynamics is commonly said to be *Cournot stable* (see Vives, *op.cit.*). If the PSNE is not unique, it is shown that the best-reply dynamics will converge to a *set* of pure strategy Nash equilibria (which is the best one can hope for in a situation where the best-reply dynamics need not converge at all). This may be called *weak Cournot stability*.

Several corollaries illustrate how these observations can be applied constructively. It readily follows, for example, that any supermodular game with single-valued best-replies and a unique PSNE, is a best-reply potential game. Of course there is also the immediate consequence that stability is established for many frequently studied games. A case in mind is aggregative games with strategic substitutes (Dubey et al. (2006), Jensen (2007), Acemoglu and Jensen (2008)). Thus we get a direct generalization of Cournot's original stability result, to Cournot oligopoly with any number of firms: If best-replies are single-valued and decreasing, a unique equilibrium is Cournot stable (and if the equilibrium is not unique, sequential paths will converge to a set of Cournot equilibria).

2 Results

A game is here a list $\Gamma = (r_i, S_i)_{i\in\mathscr{S}}$ where $\mathscr{S} = \{1,\dots,I\}$ is a finite set of players, S_i the strategy set, and $r_i : S_{-i} \equiv \prod_{j\neq i} S_j \to S_i$ the best-reply map of player i. The joint strategy set $S = \prod_i S_i$ is merely assumed to be a compact metric space (in particular, the dimension is not assumed to be finite and strategy sets are not assumed to be convex).

Γ is a *best-reply potential game* (Voorneveld (2000)) if there exists a function $P: S \to \mathbb{R}$ (the potential) such that for all $i \in \mathscr{S}$:

$$\{r_i(s_{-i})\} = arg \max_{s_i \in S_i} P(s_i, s_{-i}) \text{ , for all } s_{-i} \in S_{-i} \tag{1}$$

In words, a game is a best-reply potential game if each player instead of maximizing his payoff function can maximize the "common objective" P. Throughout, the best-reply potential is assumed to be a continuous function (see remark 2.1).

A sequence $(s^t)_{t=0}^{\infty}$ in S is a *sequential best-reply path* if players move in turn and always to a best-reply, *i.e.*, if at all t, there is some i (the player whose turn it is to move) such that $s_{-i}^t = s_{-i}^{t+1}$ and $s_i^{t+1} = r_i(s_{-i}^t)$. Since it is possible that the player who moves chooses to stay with the previous strategy $s_i^{t+1} = s_i^t = r_i(s_{-i}^t)$, the infinite repetition of a PSNE is a sequential best-reply path. A sequential best-reply path is *admissible* if whenever I successive periods have passed, all I players have moved. Nothing interesting can be said in general about paths which are not admissible.[1]

Convergence of (s^t) to a point s^* has the usual meaning with reference to the metric topology on S.[2] A sequence (s^t) *converges to the set* $A \subseteq S$ if any convergent subsequence of (s^t) has its limit in A. Formally, this means that A is the *positive* (or *omega*) limit of the sequence (s^t) (see *e.g.*, Agerwal (2000), chapter 5). We are now ready to state this note's two main results. All proofs are placed in section 3.

Theorem 1 *Let Γ be a best-reply potential game with single-valued, continuous best-reply functions and compact strategy sets. Then any admissible sequential best-reply path converges to the set of pure strategy Nash equilibria.*

Theorem 2 *Let Γ be a game with continuous best-reply functions, compact strategy sets, and a unique pure strategy Nash equilibrium $s^* \in S$. Then Γ is a best-reply potential game if and only if every admissible sequential best-reply path converges to s^*.*

Remark 2.1 *Without any continuity assumption on the potential P, the sufficiency part of theorem 2 is false. On the other hand, continuity of P strengthens the necessity part's statement. For ways of weakening the continuity assumptions on P see remark 3.1.*

[1]The trivial counterexample to any form of convergence to a PSNE is when the *same* player moves at all dates and so simply stays with the same strategy (which of course need not be a PSNE). For further discussion see Remark 1 in section 4.

[2]If S is finite the topology is discrete (because it is Hausdorff), and convergence to s^* then means that (s^t) "terminates" at s^*, *i.e.*, $s^t = s^*$ for all $t \geq T$, for some $T \in \mathbb{N}$.

The next three results are among many possible applications of the equivalence established in theorem 2. Recall that a game is *dominance solvable* if the set which remains after iterated elimination of dominated strategies is a singleton. Moulin (1984) shows that if a game is dominance solvable, its (unique) PSNE is Cournot stable. Combining with theorem 2 we get:

Corollary 1 *Any dominance solvable game with continuous best-reply functions and compact strategy sets, is a best-reply potential game.*

Any set of conditions which implies Cournot stability will, of course, equally imply that a game is a best-reply potential game.

Our next result follows directly from a combination of theorem 2 above, the corollary to Theorem 1 in Moulin (1984), and the second corollary to Theorem 5 in Milgrom and Roberts (1990). For the relevant definitions see Milgrom and Roberts (1990).

Corollary 2 *Any supermodular game with continuous best-reply functions, compact strategy sets, and a unique PSNE, is a best-reply potential game.*

Finally, let us explicitly mention a consequence that was touched upon already in the introduction. A game is *aggregative* if payoff functions take the form $\pi_i(s) = \tilde{\pi}_i(s_i, \sum_j s_j)$, $i \in \mathscr{S}$ (cf. Corchón (1994)). It has *strategic substitutes* if the best-reply functions are decreasing in opponents' strategies. It is proved in Dubey et al. (2006) that a strictly quasi-concave, aggregative game with strategic substitutes admits a continuous best-reply potential, and so we get:

Corollary 3 *Let* Γ *be an aggregative game with strategic substitutes, upper semi-continuous, strictly quasi-concave payoff functions, and compact one-dimensional strategy sets. Then any admissible sequential best-reply path converges to a set of pure strategy Nash equilibria and if the PSNE is unique, it is Cournot stable.*

The result on Cournot oligopoly stated in the introduction is a special case of this result. The observation that aggregative games are best-reply potential games is valid for a much wider class of "aggregation rules" (payoffs depending on some other function of opponents' strategies than the above *linear sum*, see Jensen (2007)).

3 Proofs

To simplify notation, define two relations on S: (i) $\tilde{s} \succ s \Leftrightarrow \exists i \in \mathscr{S}$ s.t. $[\tilde{s}_{-i} = s_{-i}$, $s_i \neq r_i(s_{-i})$ and $\tilde{s}_i = r_i(s_{-i})]$. (ii) $\tilde{s} \succeq s \Leftrightarrow \exists i \in \mathscr{S}$ s.t. $[\tilde{s}_{-i} = s_{-i}$ and $\tilde{s}_i = r_i(s_{-i})]$. Lemma 1 in Jensen (2007) gives an equivalent characterization of best-reply potential games in terms of the previous two relations (in the more general situation with best-reply correspondences). The following is a direct consequence of that Lemma when best-replies are assumed to be single-valued. The proof is therefore omitted.

Proposition 1 *The game $\Gamma = (r_i, S_i)_{i\in\mathscr{S}}$ is a best-reply potential game if and only if there exists a real-valued function, $P : S \to \mathbb{R}$ such that for all $s, \tilde{s} \in S$:*

$$\tilde{s} \succ s \Rightarrow P(\tilde{s}) > P(s) \tag{2}$$

3.1 Proof of Theorem 1

Write $\mathbf{s} = (s^t)_{t=0}^{\infty}$ for a sequential best-reply path. The *positive limit set* is $\Omega(\mathbf{s}) = \{\lim_{k\to\infty} s^k : (s^k)_{k=0}^{\infty}$ a convergent subsequence of $\mathbf{s}\}$. The following is an adaption of a standard result from Liapunov methods (see *e.g.*, Agerwal (2000)).

Lemma 1 *If S is compact and the potential P is continuous, $\Omega(\mathbf{s}) \subseteq P^{-1}(p) = \{s \in S : P(s) = p\}$ for some $p \in \mathbb{R}$.*

Proof: *For $s^{\omega} \in \Omega(\mathbf{s})$, there exists by definition a subsequence of $\mathbf{s}$, $(s^{t_l})_{l=0}^{\infty}$, such that $\lim_{l\to\infty} s^{t_l} = s^{\omega}$. $(P(s^t))_{t=0}^{\infty}$ is a non-decreasing sequence by* (2). *So $\lim_{l\to\infty} P(s^{t_l}) = \lim_{t\to\infty} P(s^t) = p < +\infty$ (where the last inequality follows from continuity of P). By continuity $P(s^{\omega}) = p$.* □

Remark 3.1 *For the conclusion of lemma* 1 *(hence for theorem 1 and sufficiency in theorem* 2*), continuity of P is not necessary. Sufficient would be: $P(s) = \lim_{n\to\infty} P(s^n)$ whenever $s^n \to s$ and the sequence satisfies: $P(s^n) \le P(s^{n+1})$ for all n. This condition is not implied by upper semi-continuity of P (although it is implied by the closely related, but stronger, notion of limsup continuity,* $\limsup_n P(s^n) = P(s)$*).*

We are now ready to prove the claim of theorem 1. Let $\mathbf{s}$ be an admissible sequential best-reply path and (s^{t_l}) a convergent subsequence, $s^{t_l} \to s$ as $l \to \infty$. So $s^{t_l} \preceq s^{t_l+1} = (r_{i(t_l)}(s^{t_l}_{-i(t_l)}), s^{t_l}_{-i(t_l)})$ for all l, with $i(t_l) \in \mathscr{S}$ the player deviating from s^{t_l}. Since the sequence is infinite and the number of players is finite, we may pick a subsequence such that for some player, say $i = 1$: $s^{t_{l_1}} \preceq s^{t_{l_1}+1} = (r_1(s^{t_{l_1}}_{-1}), s^{t_{l_1}}_{-1})$ for all l_1. As $s^{l_1} \to s$, it follows that $s^{t_{l_1}}_{-1} \to s_{-1}$. Hence, by continuity of the best-reply function $s^{t_{l_1}+1} \to (r_1(s_{-1}), s_{-1})$. Note that $\{s, (r_1(s_{-1}), s_{-1})\} \subseteq \Omega(\mathbf{s})$, so by lemma 1, $P(s) = P(r_1(s_{-1}), s_{-1})$. But then, because of (2), it must be the case that $r_1(s_{-1}) = s_1$. So in fact, $s^{t_{l_1}+1} \to s$. Now, $(s^{t_{l_1}+1})$ is itself a convergent subsequence of (s^t), so we may begin over again considering from the beginning $(s^{t_{l_1}+1})$. At every date t_{l_1}, player 1 moves by construction; and the same player can never move twice in a row on an admissible path. Thus we get another player, say $i = 2$, such that $r_2(s_{-2}) = s_2$ and $s^{(t_{l_1}+1)_2+1} \to s$ where $s^{(t_{l_1}+1)_2}$ is a subsequence of $s^{t_{l_1}+1}$. We can repeat this argument for all players, because the path is admissible, and conclude that $r_i(s_{-i}) = s_i$ for all i. So the limit of s^{t_l} is a fixed point of the best-reply function $r = \prod_{i\in\mathscr{S}} r_i$, *i.e.*, a PSNE. As this was the limit of an arbitrary convergent subsequence of $\mathbf{s}$, we conclude that all elements in $\Omega(\mathbf{s})$ are PSNE.

3.2 Proof of Theorem 2

Sufficiency in theorem 2 is an immediate consequence of theorem 1: If there is a unique fixed point, $s^* \in S$, then every subsequence of an admissible sequence (s^t) converges to the same point, s^*. The sequence is consequently Cauchy, and because the space is complete (it is compact by assumption), it consequently converges.

We now turn to necessity. Let $\mathscr{F}(s)$ denote the set of all admissible sequential best-reply paths starting at s. $\mathscr{F}(s)$ is equipped with the topology of pointwise convergence. As may be verified, $\mathscr{F}(s)$ is closed. The correspondence which maps s to $\mathscr{F}(s)$ is continuous because the best-reply maps are continuous. From the given metric $d : S \times S \to \mathbb{R}_+$ on S define a real-valued function:

$$D(s,\tilde{s}) = \sup_{\mathbf{s}\in\mathscr{F}(s),\tilde{\mathbf{s}}\in\mathscr{F}(\tilde{s})} \sup_t d(s^t,\tilde{s}^t)$$

Notice that (i) $D : S \times S \to \mathbb{R}_+$ is continuous, (ii) if $\mathbf{s}$ is a sequential best-reply path, then $D(s^{t+1},s^{t+2}) \leq D(s^t,s^{t+1})$ for all t and $D(s^t,s^{t+1}) \to 0$, (iii) if $s \neq \tilde{s}$ then $D(s,\tilde{s}) > 0$. Our candidate best-reply potential function $P : S \to \mathbb{R}$ is:

$$P(s) = -\sup_{\mathbf{s}\in\mathscr{F}(s)} [\sum_{t=0}^{\infty} \delta^t D(s^{t+1},s^t)] \tag{3}$$

where $0 < \delta < 1$. Clearly P is a continuous and non-positive function with maximum value 0 assumed at the PSNE s^* (this is because when s^* is a PSNE, $\mathscr{F}(s^*) = (s^*,s^*,\ldots)$). To show that it is indeed a best-reply potential, we verify the conditions in lemma 1. So pick $\tilde{s} \succeq s$, $s \neq \tilde{s}$ (if $s = \tilde{s}$ there is nothing to prove), and set $\tilde{\mathscr{F}} = \{\mathbf{s} \in \mathscr{F}(s) : s^1 = \tilde{s}\}$.

$$P(s) \leq -[D(\tilde{s},s) + \sup_{\mathbf{s}\in\tilde{\mathscr{F}}} \sum_{t=1}^{\infty} \delta^t D(s^{t+1},s^t)] =$$

$$-D(\tilde{s},s) - \delta \sup_{\mathbf{s}\in\mathscr{F}(\tilde{s})} \sum_{t=0}^{\infty} \delta^t D(s^{t+1},s^t)] = -D(\tilde{s},s) + \delta P(\tilde{s})$$

We now prove that $-D(\tilde{s},s) + \delta P(\tilde{s}) < P(\tilde{s})$, which implies that $P(s) < P(\tilde{s})$ and therefore completes the proof. Rewrite as, $(\delta - 1)P(\tilde{s}) < D(\tilde{s},s)$, and notice that because D is non-expansive in the sense of (ii) above, we always have that $(\delta-1)P(\tilde{s}) = (1-\delta)|P(\tilde{s})| \leq (1-\delta)\frac{D(\tilde{s},s)}{1-\delta} = D(\tilde{s},s)$, with equality throughout if and only if $D(\tilde{s},s) = D(s^{t+1},s^t)$ for all t (here the path is the maximizer in (3)). But since $D(s^t,s^{t+1}) \to 0$ as $t \to \infty$ (the path converges to a fixed point, again by (ii) above), this can never be the case.

4 Concluding Remarks

This note's results do not stand and fall with the realism of sequential best-replies as a theory of "learning". Convergence of sequential paths can be seen as a technical condition, and several corollaries above are free from all reference to Cournot dynamics. Nonetheless, Cournot dynamics (whether sequential or simultaneous) also has its virtues. One is that it is easy to implement on a computer. Thus one can *compute* a pure strategy Nash equilibrium via this method (by iterating), and may also *reject* the hypothesis that a game is a best-reply potential game (by finding a sequential path that does not converge).[3]

Looking at proposition 1, it should be noticed that (2) can be read as saying that the game's best-reply dynamics admit the strict Liapunov function $V = -P$ (see *e.g.*, Agerwal (2000)). Does this mean that game theorists when studying best-reply potential games have merely rediscovered Liapunov theory ? The reasonable answer is *no*, for the reason that best-reply dynamics does not correspond to such standard dynamical systems as Liapunov methods are developed to address. This is also evident from the proof of theorem 2 which, lemma 1 aside, has no direct relationship with standard Liapunov techniques. Nonetheless, the close relationship with Liapunov methods perhaps warns us not to overinterpret the possibility of reformulating a game so that players maximize the same function rather than their individual payoffs.

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[3] See Echenique (2004), especially footnote 5. Echenique's paper is no doubt better written than mine, and is subject to the exact same objections, and the same response.

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