Lecture Notes, Econ 320B. Set # 4.

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1 Producers

We now introduce firms into our economy. There are $J \in \mathbb{N}$ firms and $\mathscr{J} = \{1,\ldots,J\}$ denotes the set of firms. As before there are $n \in \mathbb{N}$ goods. A **production plan** for firm $j \in \mathscr{J}$, denoted by y^j is an n-dimensional vector, *i.e.*, $y^j \in \mathbb{R}^n$. The important thing to understand is the following convention: If $y_k^j > 0$ (where k is the index of some good), then the firm *produces* the good (there is positive output). If $y_k^j < 0$, then the firm *uses* the good for production (it is an input/there is "negative output").

Example 1 *Take $n = 2$. Then $y^j = (1,-1)$ means that the firm produces one unit of the first good by means of one unit of the second good.*

A firm j's **production possibility set** (or more briefly, **production set**) denoted Y^j is the set of all the production plans that are technologically feasible for the firm. It is clear that $Y^j \subseteq \mathbb{R}^n$.

Example 2 *Take $n = 2$ and $Y^j = \{(1,-1)\}$. This somewhat "perverse" example means that the firm has only one possible choice, namely to produce one unit of the first good by means of one unit of the second good. If $Y^j = \{(0,0),(1,-1)\}$ the firm now has two possible production plans: The one just mentioned and "shutting down" (producing nothing by means of nothing).*

Example 3 *Take $n = 2$ and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a function (for example we could have $h(z) = z^\alpha$ where $\alpha > 0$). Then take:*

$$Y^j = \{(y_1^j, y_2^j) \in \mathbb{R}_+ \times \mathbb{R}_- : 0 \le y_1^j \le f(-y_2^j), -b \le y_2^j \le 0\}$$

Here (i) the firm can use no more than b units of the second good, and (ii) Given y_2^j (the input which is negative!) the firm can produce no more than $f(-y_2^j)$ units of the first good. This is like a production function known to you from macro. The only difference here is that we need to take account of the convention that inputs are negative numbers.

Assumption 5.2.

1. $\mathbf{0} \in Y^j \subseteq \mathbb{R}^n$ *[Possibility of inaction]*
2. Y^j *is closed and bounded [Compactness]*

3. *Y^j is strongly convex. That is, for all $y, \tilde{y} \in Y^j$, $y \neq \tilde{y}$ and all $0 < t < 1$, there exists some $\bar{y} \in Y^j$ such that $\bar{y} \geq ty + (1-t)\tilde{y}$ and equality does not hold.*[1] *[Strong convexity]*

Remark 1 *Compactness is much stronger than we need for any of our results, as is by the way strong convexity. It would be quite sufficient to assume that Y^j is merely closed (so not necessarily bounded) and that Y^j is convex. Assumption 5.2. makes life very easy, though. In particular, strong convexity ensures that firms have a unique profit maximizing production plan given prices $p \gg 0$. So we get nice supply functions; which we would not necessarily get with only convexity. Still, keep in mind that strong convexity rules out a very important case, namely constant returns to scale. But as mentioned, all our important results are true under convexity only, hence true under constant returns to scale.*

We are now ready to define the **objective of the firm** in a competitive economy which is to **maximize profits**. This gives rise to the following problem (called the firm's **profit maximization problem** or PMP):

(**PMP**) $\max_{y^j \in Y^j} py^j$

As I've mentioned to you several times, py^j is equal to $\sum_{k=1}^n p_k y_k^j$ so we can also write the PMP as:

(**PMP**) $\max_{y^j \in Y^j} \sum_{k=1}^n p_k y_k^j$

The thing to notice is that because inputs are negative numbers, the sum being maximized here consists of positive and negative entries. If, for example, $n = 2$ and $y_1^j > 0$ and $y_2^j < 0$, profits will be $p_1 y_1^j + p_2 y_2^j$ where $p_1 y_1^j \geq 0$ (the revenue) and $p_2 y_2^j \leq 0$ (the cost).

In JR p.207 you'll find a theorem (theorem 5.9) which mirrors our result on consumers. This says: *If Y^j satisfies Assumption 5.2., then for all $p \gg 0$, the solution to the firm's profit maximization problem exists and is unique.* Thus we can define $y^j(p)$ as **the firm's profit maximizing production plan given** $p \gg 0$ ($y^j(p)$ is also referred to as the **supply function**). Notice how this is very similar to when we defined consumer's demand functions from the UMP (in that case uniqueness followed from strict quasi-concavity of utility functions). Moreover, *the (vector-valued) function $y^j(p)$ will be continuous in p on $\mathbb{R}^n_{++}$.* Finally, the **profit function**:

[1] The inequality in $\bar{y} \geq ty + (1-t)\tilde{y}$ is *coordinatewise*. So this is the same as writing that $\bar{y}_k \geq ty_k + (1-t)\tilde{y}_k$ for all $k = 1, \ldots, n$.

$$\Pi^j(p) = py^j(p)\,,$$

will be continuous in p on $\mathbb{R}^n_+$.

2 Consumers in Private Ownership Economies

We now take a second look at consumers. Recall that in the exchange economy setting, the income $m^i(p)$ of a consumer given prices p and initial endowments e^i was:

$$m^i(p) = pe^i$$

When there are firms in an economy, consumers still earn income from their initial resources but they also earn income from **ownership shares** in the firms. Specifically, a consumer i is now assumed to hold a share of ownership in firm j denoted by θ^{ij}. θ^{ij} is a number between 0 and 1, $0 \leq \theta^{ij} \leq 1$. If, for example, $\theta^{7,5} = 0.5$ (notice the comma here which sort of is necessary to separate the i from the j in general when we insert actual numbers); this means that consumer number 7 owns $0.5 = 50\%$ of firm number 5. So this consumer is then entitled to fifty percent of firm 5's profit.[2]

Every consumer is consequently described by a sequence of ownership shares $\theta^i = (\theta^{i1}, \ldots, \theta^{iJ})$. Many of these could be zero, of course, in fact they might all be (this means then that this specific consumer doesn't own any shares in any firms). In a **private ownership economy** consumers collectively own the firm, that is to say that for each $j \in \mathscr{J}$:

$$\sum_{i=1}^{I} \theta^{ij} = 1$$

So take the sum of all consumers' ownership shares in a given firm and you get 1 (= 100%). This is indeed private ownership as opposed to a situation where consumers own less than 100% of the shares and the government owns the rest (like the UK financial system here anno 2009).

Now, given the price vector $p \gg 0$, firm j will earn profit $\Pi^j(p)$. The income of consumer i is therefore:

[2] If there is any profit. The firm might not earn any profit of course. But notice that because of the possibility of inaction assumption, a firm *never* earns negative profit (it never looses money). This is because it can always choose $y^j = 0$ which yields profit $p0 = 0$.

$$m^i(p) = pe^i + \sum_{j=1}^{J} \theta^{ij} \Pi^j(p)$$

At the lectures, I'll speak a little of Theorem 5.12 in JR (page 209), so if you didn't go to the lecture be sure to read this at home. The main point is that there exists a unique solution to each consumer i's utility maximization problem:

$$\begin{aligned} &\max\ u^i(x^i) \\ &\text{s.t.} \begin{cases} px^i \leq m^i(p) \\ x_k^i \geq 0 \text{ for } k = 1, \ldots, n \end{cases} \end{aligned} \tag{1}$$

Solving this really is no different from solving the UMP in an exchange economy. The demand function is now most conveniently written as:

$$x^i(p, m^i(p))$$

And (also part of Theorem 5.12), this will be continuous in p which one uses when existence is proved (which we, however, shall not do).

3 Market Clearing and Existence of WE

Markets' clear (are in equilibrium) in an economy with production if aggregate demand equals aggregate supply. In an exchange economy, aggregate supply is equal to the sum of the consumers' initial resources. With production, aggregate supply equals this *plus* the sum of firms' supplies. A **Walrasian equilibrium** is a price vector $p^* \gg 0$ such that markets clear:

$$\sum_{i \in \mathscr{I}} x^i(p^*, m^i(p^*)) = \sum_{i \in \mathscr{I}} e^i + \sum_{j \in \mathscr{J}} y^j(p^*)$$

The associated **Walrasian Equilibrium Allocation** or **WEA** is:

$$(x(p^*), y(p^*)) = ((x^1(p^*, m^1(p^*)), \ldots, x^I(p^*, m^I(p^*))), (y^1(p^*), \ldots, y^J(p^*)))$$

We are now in a position to ask whether things make sense at all (I'm referring here to the long speech I held about existence results!). And under the assumptions we have introduced so far, this is fortunately the case:

Theorem 5.13 (**Existence of WE with Production**) Consider a private ownership economy with production $(u^i, e^i, \theta^{ij}, Y^j)_{i\in\mathscr{I}, j\in\mathscr{J}}$. If each u^i satisfies assumption 5.1., each Y^j satisfies assumption 5.2., and $y + \sum_{i\in\mathscr{I}} e^i \gg 0$ for some aggregate production vector [this means that $y = \sum_{j\in\mathscr{J}} y^j$ where $y^j \in Y^j$ for all j], then there exists a Walrasian equilibrium.