Equilibria in the War of Attrition

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1 Introduction

This paper will prove that in a two-player, complete information War of Attrition there are three equilibrium points. One of these equilibrium points varies based on the value attached to the pay-off by the players and the cost of fighting in the game. The remaining two points are the same regardless of the values and costs. These equilibrium points can be used to determine the optimal strategy for a player in the game.

The results in this paper are based on Peter Burton's paper *A Note: Complete Information Wars of Attrition and the Generalised Nash Bargaining Solution [2001]*.

2 Definitions and Assumptions

There are three key assumptions that will be made in the model:

1. For simplicity the game is played by only two players, denoted by 1 and 2[1]. The players are rational and seek to maximise their gains and minimise their losses.
2. The players have complete information, meaning that the each player knows everything about the game including how the other player values the pay-off, the strategies available to the other player and the costs incurred.
3. The game is stationary, meaning that past time periods have no effect on player's decisions, strategies or outcomes in any subsequent time period. This allows the model to be representative of the whole game whilst analysing only one time period.

The two players fight over a single indivisible prize of values v_1 and v_2 respectively. The game takes place over time and for each period of time the two players must simultaneously and independently choose to drop out or to

[1]The game is possible with any number of players as demonstrated in Klemperer and Bulow's paper *The Generalised War of Attrition [1999]*.

continue fighting, which will cause them to incur an unrecoverable cost, c. If both players choose to fight the game progresses into the next time period and both players incur the associated cost.

The game ends when at least one player chooses to drop out - if one player chooses to drop out during a given time period he will lose the sum of the cost of fighting for all previous periods of time and the other player will win the prize. If both players drop out in the same round, neither wins the prize and both lose their unrecoverable costs.

The probability of a player dropping out in any given time period is denoted as p_1 and p_2, where $p = 1$ means dropping out is a certainty and $p = 0$ means the player will not drop out that time period.

An important concept in the model is Nash equilibrium (or just equilibrium), which refers to a situation in which the decisions of the players mean that no player could do better given the choice of his opponent. Equilibrium is very important when considering the optimal strategy for a player.

3 Main Result

The probability of a player winning the pay-off in a given time period is the probability of the other dropping out (so long as the first player does not also drop out, in which case the game would over). The expected gain for any time period for the player is therefore the probability of winning that time period multiplied by the value that would be gained from winning:

$$\textit{expected gain} = p_2 v_1$$

As a cost is incurred for each time period, the expected net benefit is the expected gain minus the cost:

$$\textit{expected net benefit} = p_2 v_1 - c$$

If the expected net benefit to a player is positive he will always choose to fight at the start of a round (in order to maximise his gain). Conversely if the expected net benefit is negative he will always choose to drop out (in order to minimise his loss):

$$p_1 = 0 \text{ when } p_2 v_1 - c > 0$$
$$p_1 = 1 \text{ when } p_2 v_1 - c < 0$$

When the expected net benefit for a player is 0 he has nothing to gain or lose from choosing to fight. He therefore is indifferent between dropping out and fighting and the probability of dropping out could be anything between 0 and 1:

$$0 < p_1 < 1 \text{ when } p_2 v_1 - c = 0 \text{ or } p_2 v_1 = c$$

This equation can be algebraically manipulated to show that the player is indifferent between dropping out and fighting when the ratio of cost per time period to the value placed on the pay-off by the player is equal to the probability of his opponent dropping out:

$$p_2 = \frac{c}{v_1}$$

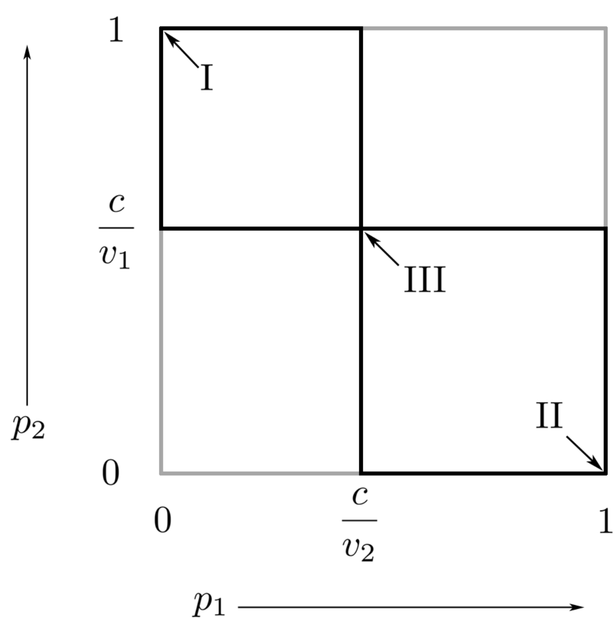
As the game involves only two players the above observations hold true for both players - the two players could be swapped with no effect. This leads to three equilibria points in the game being established:

Equilibrium	p_1	p_2
I	1	0
II	0	1
III	$\frac{c}{v_2}$	$\frac{c}{v_1}$

I. Player 1 certainly drops out and player 2 certainly fights - player 2 wins.

II. Player 2 certainly drops out and player 1 certainly fights - player 1 wins.

III. Both players are indifferent between dropping out and fighting. The probability that a player drops out is higher the lower his opponent values the pay-off. Neither player necessarily wins, but one player will be more likely to do so if his opponent values the pay-off more than him.



The three equilibria are illustrated on the above diagram. The x-axis shows the probability of player 1 dropping out and the y-axis shows the probability of player 2 dropping out. Points I and II are fixed in every possible game, however point III can be at any point between $(0, 0)$ and $(1, 1)$ depending on the value of c, v_1 and v_2 in a given game. If v exceed c the equilibrium point would be outside of the diagram and the probability would be greater than 1; this represents the fact that if the cost of fighting in a single time period is greater than the value placed on the pay-off the player will certainly not fight.

4 Discussion

The game is stationary and the equilibrium points I and II show that one of the two players will always drop out in a given time period if the expected net benefit for the time period for either player is negative. This leads to an interesting implication of the model - in the complete information War of Attrition the game will be over in the first time period due to one player dropping out if the equilibrium point is either I or II[2]. In a situation where the expected net benefit is not 0 a players optimal strategy is therefore to play the equilibrium point available to him (I or II) as it minimises losses and maximises gain (however the gain could be zero if $p = 1$ for a player).

Equilibrium point III is of particular interest due to counter-intuitive result it presents. It would be expected that the more a player values the pay-off the more likely he is to win, however this is not the case. A full explanation of this phenomena is beyond the scope of this paper however Levin's paper *Wars of Attrition [2004]* looks at it in more in depth. If the players equilibrium for a game is at point III and the game were to be played with both players randomly choosing to fight or drop out at the start of each time period based on their values of p, the player with the lower p is less likely to drop out and more likely to win. To minimise losses the player with the higher value of p may therefore choose to drop out immediately.

The reliability of this model when predicting real-life situations is heavily influenced by the plausibility of the assumptions being made. Issues are most likely to arise from the second assumption of complete information - as the War of Attrition is often used to model competition in markets complete information is unlikely as competitors would aim to hide their strategies and valuations from each other. Despite this, it is possible that market research and past behaviour of competitors could be used to predict the information lost when there is not complete information, meaning the assumption is not entirely unrealistic.

[2] A similar observation is discussed at length in Myatt's paper *Instant Exit from the War of Attrition [1999]*.