Asymmetric play in a linear quadratic differential game with bounded controls

Colin Rowat

Department of Economics, University of Birmingham, Edgbaston B15 2TT, (t) +44 121 414 3754, (f) + 44 121 414 7377

Abstract

This paper uses computational techniques to identify the Markov perfect equilibria in a two agent linear quadratic differential game with bounded controls. No evidence is found asymmetric equilibria when the agents are symmetric or of non-linear equilibria when the agents are asymmetric. This suggests that the standard continuum result for identical agents is not robust and that non-linear strategies are not of general interest in the analysis of linear quadratic differential games. The techniques presented here are applicable to a broader class of differential games as well.

Key words: linear quadratic differential game, Markov perfect Nash equilibrium, non-linear strategies, numerical methods

1 Introduction

This chapter extends the model of Rowat (2002) by allowing asymmetric play in a linear quadratic differential game with bounded controls. The body of this paper works with identical agents; Appendix A considers non-identical agents. While identical agents are typically assumed to play symmetrically they are assumed to play asymmetrically here for two reasons.

Email address: `c.rowat@bham.ac.uk` (Colin Rowat).

[1] My deepest thanks are due to Jim Mirrlees for his careful supervision. The comments of Luca Anderlini, Chris Harris, Ngo Van Long, Hamish Low, David Newbery, Gerhard Sorger, Shunichi Tsutsui and Jean Pierre Vidal are also apppreciated. The support of Marika Asimakopulos, Jayasri Dutta, the Overseas Research Students scheme and the Cambridge Commonwealth Trust have also been vital. The above are absolved of responsibility for any remaining errors.

First, exposition is simplified while still introducing all the numerical techniques required for the appendix' asymmetric analysis. Second, the special case of identical agents' symmetric play may be tested against the results in Rowat. To facilitate this test, a linear quadratic game is explored here. The techniques presented, though, are applicable to more general differential games as well.

Technically, the difference between this paper and its predecessor, Rowat (2002), is that the previous single ordinary differential equation is now replaced by a system of two ordinary differential equations. As systems of differential equations are less likely to yield analytical solutions than are ordinary differential equations, numerical techniques are adopted here. While less transparent than are analytical solutions, numerical techniques allow solution of more complicated problems, such as those with non-linear equations of motion or asymmetric agents.

Against this advantage, the numerical techniques implemented here are unable to consider discontinuous strategies. Dockner and Sorger's model (Dockner and Sorger, 1996) with discontinuous MPE strategies shows that these should not be dismissed *a priori*. Their model differs from the present, though, in at least two important ways: there is no glut point in consumption and increases in agents' controls decrease the state variable. Neither of these features, which are required for their discontinuous MPE strategies, are present here.

Numerical analysis proceeds by integrating over the state space from a grid of initial conditions. The conditions presented in Rowat (2002) are applied to rule out strategies, refining the candidate set.

In general, a grid of initial conditions will not find isolated MPE strategies unless it is adapted to finding them. In an attempt to find these isolated strategies, an adapted second grid of initial conditions is therefore developed, based on the following observation: in Rowat, the isolated linear MPE strategy was identified as a singular solution because it intersected another linear candidate solution, uniquely among all the candidates. As this suggests a relationship between MPE and singularities, the singularity locus is identified for this case of asymmetric play and the strategies through it calculated. For identical agents, the locus is based on a conic section; otherwise the locus is more complicated.

Section 2 presents the model. Section 3 contains definitions and conditions required to test whether a solution to the differential equation system is an MPE. Section 4 examines the singularity locus while Section 5 discusses the coding and execution. Section 6 presents the results of the numerical analysis and Section 7 concludes. Appendix A extends analysis to the case of non-identical agents.

2 The linear quadratic model

Consider two agents, indexed by $i = 1, 2$. Each chooses its respective $x_i \in \Re_+$, thus controlling a state variable, $z \in Z \equiv \Re_{++}$, which evolves according to the linear differential equation of motion

$$\dot{z} = x_1(t) + x_2(t) - \beta z(t) \text{ s.t. } z(0) = z > 0; \tag{1}$$

where $\beta \in \Re_+$ is a constant term. The strict inequality simplifies consideration of activity around the state space's lower bound.

Grant the agents identical quadratic instantaneous utility functions

$$u(x_i, z) = -(x_i - \xi)^2 - \nu(z - \zeta)^2; \tag{2}$$

where ν, ξ and ζ are positive real constants. Instantaneous utility is therefore concave in both control and state.

Agents' intertemporal objective functions are of the form

$$\int_0^\infty e^{-\delta t} u(x_i(t), z(t))\, dt; \tag{3}$$

where $\delta \in \Re_{++}$ is a discount rate and t is time, assumed, for calibration purposes, to be in years.

The linear equation of motion and quadratic objective function together define a *linear quadratic game*. The example motivating this one is a greenhouse gas emissions problem. In this light, x_i may be thought of as nation i's greenhouse gas emissions, produced in a fixed ratio to and as a byproduct of national production, z the atmospheric stock of greenhouse gasses and β the decay rate.

Agents' Bellman equations are of the form

$$\delta V_i(z) = \max_{x_i \geq 0} \left\{ -(x_i - \xi)^2 - \nu(z - \zeta)^2 + V_i'(z)(x_1 + x_2 - \beta z) \right\}, i = 1, 2; \tag{4}$$

when the value function is piecewise $\mathcal{C}^1$, an assumption maintained throughout. Its first order conditions are

$$x_i^* = \max\left\{0, \xi + \frac{V_i'(z)}{2}\right\}, i = 1, 2. \tag{5}$$

As the objective functions are concave, x_i^* is unique and a maximiser; substitute it into equation 4 for

$$\delta V_i(z) = -(x_i^* - \xi)^2 - \nu(z - \zeta)^2 + V_i'(z)(x_i^* + x_{-i} - \beta z), i = 1, 2; \tag{6}$$

and where $-i$ indicates the agent that is not agent i.

Although x_i^* is unique, solutions to differential equation 6 are not. Therefore, solutions to differential equation 6 are not generally solutions to maximisation problem 4. Denote a solution to equation 6 by $W_i\left(\right)$; call this a *candidate value function*; let $\mathcal{W}_i$ be the family of solutions to equation 6. Therefore $V_i \in \mathcal{W}_i$.

When the candidate value function is twice differentiable, differentiating equation 6 with respect to z yields, with some manipulation

$$\begin{aligned} & w_i'\left(z\right)\left(x_1^* + x_2^* - \beta z\right) - 2x_i^{*\prime}\left(x_i^* - \xi\right) \\ & = \left(\beta + \delta - x_1^{*\prime} - x_2^{*\prime}\right) w_i\left(z\right) + 2\nu\left(z - \zeta\right), i = 1, 2. \end{aligned} \tag{7}$$

when $w_i \equiv W_i'$.

The following assumes this second differentiation to be valid; points at which it is not, called *non-invertible*, will be identified numerically.

As either $x_i^* > 0$, the interior of the action space, or $x_i^* = 0$, the action space's corner, there are three possible scenarios for each $z \in Z$: both agents play in the interior, both play on the corner, or one plays in the interior and the other on the corner. These are now explored.

2.1 Both agents interior

In this case, $x_i^* > 0 \forall i = 1, 2 \Rightarrow x_i^* = \xi + \frac{1}{2} w_i\left(z\right), x_i^{*\prime} = \frac{1}{2} w_i'$ and equations 7 become the differential equation system

$$\begin{bmatrix} g & \frac{1}{2}w_1 \\ \frac{1}{2}w_2 & g \end{bmatrix} \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} \left(\beta + \delta\right) w_1 + 2\nu\left(z - \zeta\right) \\ \left(\beta + \delta\right) w_2 + 2\nu\left(z - \zeta\right) \end{bmatrix}; \tag{8}$$

where

$$\dot{z} = g\left(\boldsymbol{w}, z\right) \equiv \frac{1}{2}\left(w_1 + w_2\right) - \beta z + 2\xi. \tag{9}$$

A special case of this is

$$w'\left(z\right) = \frac{\left(\beta + \delta\right) w + 2\nu\left(z - \zeta\right)}{\frac{3}{2}w - \beta z + 2\xi}; \tag{10}$$

the symmetric play ($w \equiv w_1 = w_2$) equation analysed in Rowat (2002).

2.2 One agent interior, the other cornered

Assume without loss of generality that agent i has cornered. Now $x_j^* > 0, x_i^* = 0 \Rightarrow x_j^* = \xi + \frac{1}{2}w_j(z), x_j^{*\prime} = \frac{1}{2}w_j'; x_i^{*\prime} = 0$ so that equations 7 produce the differential equation system

$$\begin{bmatrix} h & 0 \\ \frac{1}{2}w_i & h \end{bmatrix} \begin{bmatrix} w_j' \\ w_i' \end{bmatrix} = \begin{bmatrix} (\beta+\delta)\, w_j + 2\nu\,(z-\zeta) \\ (\beta+\delta)\, w_i + 2\nu\,(z-\zeta) \end{bmatrix}; \tag{11}$$

where

$$\dot{z} = h(\boldsymbol{w}, z) \equiv \frac{1}{2}w_j - \beta z + \xi. \tag{12}$$

As the first equation in system 11 is independent of w_i and of similar form to equation 10 it is similarly solvable for:

$$K_j = |w_j - c - s^c(z-d)|^{\gamma^c} \left|w_j - c - s^d(z-d)\right|^{\gamma^d};$$

where

$$c \equiv 2\nu \frac{\beta\zeta - \xi}{\beta(\beta\delta) + \nu}; \tag{13}$$

$$d \equiv \frac{\xi(\beta+\delta) + \nu\zeta}{\beta(\beta+\delta) + \nu} > 0; \tag{14}$$

$$\left\{s^c, s^d\right\} \equiv \delta \pm \sqrt{\delta^2 + 4\nu}, \text{ s.t. } s^c > 0 > s^d;$$

$$\gamma^c = -\frac{2\beta - s^c}{s^d - s^c}; \tag{15}$$

$$\gamma^d = \frac{2\beta - s^d}{s^d - s^c} < 0; \tag{16}$$

and K_j is a constant of integration.

Solving the first equation in system 11 does not seem to allow an analytical solution for $w_i(z)$.

2.3 Both agents cornered

When both agents play on the corner $x_i^* = 0 \forall i = 1,2 \Rightarrow x_i^{*\prime} = 0$ and equations 7 produce the differential equation system

$$w_i' = -\frac{(\beta+\delta)\, w_i + 2\nu\,(z-\zeta)}{\beta z}, i = 1,2;$$

whose solution is

$$w_i(z) = K_i z^{-\frac{\beta+\delta}{\beta}} + 2\nu\left(\frac{\zeta}{\beta+\delta} - \frac{z}{2\beta+\delta}\right), i = 1, 2; \tag{17}$$

where K_i is a constant of integration.

The following lemma provides a sufficient condition for the system to remain in the cornered scenario once reaching it:

Lemma 1 *Let $\tilde{z}$ be the least z satisfying $x_i(z) = 0$ and $\hat{z} > \tilde{z}$ that satisfying $x_j(z) = 0$. A sufficient condition for $x_i(z) = x_j(z) = 0 \forall z > \hat{z}$ is that*

$$w_i(\hat{z}) \geq \frac{2\nu}{\beta+\delta}(\zeta - \hat{z}).$$

PROOF. At $\hat{z}, w_i(\hat{z}) \leq w_j(\hat{z})$. By equation 17, then, $K_i \leq K_j$. For the system to remain cornered it is sufficient that $w_i'(z), w_j'(z) \leq 0 \forall z > \hat{z}$. Differentiation of equation 17 converts this requirement into

$$K_j \geq K_i \geq -\frac{2\beta\nu}{(\beta+\delta)(2\beta+\delta)} z^{\frac{2\beta+\delta}{\beta}} \leq 0, \forall z > \hat{z}.$$

The inequality in K_j is thus automatic if that in K_i holds. That in K_i holds if it holds at $z = \hat{z}$ as K_i is fixed but the RHS decreases in z. Isolating K_i (as determined at $\hat{z}$) in equation 17 and substituting into the inequality produces

$$\left[w_i(\hat{z}) - 2\nu\left(\frac{\zeta}{\beta+\delta} - \frac{\hat{z}}{2\beta+\delta}\right)\right]\hat{z}^{\frac{\beta+\delta}{\beta}} \geq -\frac{2\beta\nu}{(\beta+\delta)(2\beta+\delta)}\hat{z}^{\frac{2\beta+\delta}{\beta}}.$$

Some manipulation produces the result.

3 Conditions for MPE

As noted above, solutions to systems 8, 11 and 17 are non-unique and generally do not support MPE. This section therefore establishes conditions for the assessment of particular solutions based on those presented in Rowat (2002).

First, some terminology is defined.

Definition 2 *A* system of differential equations *is*

$$\boldsymbol{A}(\boldsymbol{s})\boldsymbol{s}' = \boldsymbol{f}(\boldsymbol{s}); \tag{18}$$

where $\boldsymbol{s}$ is an n-vector dependent on its n^{th} element, the independent state variable; $\boldsymbol{A}(\cdot)$ is an $n \times n$ matrix, $\boldsymbol{s}' \equiv \left[\frac{ds_1}{ds_n}, \ldots, \frac{ds_n}{ds_n}\right]^\top$ the n-vector of derivatives and $\boldsymbol{f}(\cdot)$ is an n-vector.

As the elements of $\boldsymbol{A}$ and $\boldsymbol{f}$ need not be continuous, equation 18 is sufficiently general to address transition between regimes (e.g. systems 8, 11 and 17).

In what follows, $\boldsymbol{s}$ will be interpreted either as

$$\boldsymbol{s} = [w_1(z), w_2(z), z]^\top;$$

or as

$$\boldsymbol{s} = [w_1(z(t)), w_2(z(t)), z(t), t]^\top.$$

This latter, more complicated interpretation is used in Section 4 as systems 8, 11 and 17 are autonomous in t but not in z. Autonomy in t allows Taylor expansion about $t_0 = 0$, simplifying many equations without loss of generality.

Definition 3 *A* path*, $\boldsymbol{s}$, is a solution to system 18.*

Definition 4 *The point $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$* lies on path $\boldsymbol{s}$ *if $\boldsymbol{s} = \boldsymbol{\sigma}$ for some s_n in the state's domain.*

Three types of points are now defined: non-invertible points at which a path ceases to be a function by 'doubling back' through the domain; singular points lying on more than one path; and regular points.

Definition 5 *The point $\boldsymbol{\sigma}$ is*

(1) a non-invertible point *of system 18 if it lies on a path $\boldsymbol{s}$ such that:*
   (a)
$$\left.\frac{\partial s_i}{\partial s_n}\right|_{\boldsymbol{\sigma}} = \pm\infty$$
   for some coordinate i; and
   (b) $\exists \delta > 0$ s.t. path $\boldsymbol{s}$ is not defined at either of $\{\sigma_i - \varepsilon, \sigma_i + \varepsilon\}$ $\forall \varepsilon$ s.t. $\delta > \varepsilon > 0$.

(2) a singular point *of system 18 if it lies on at least two distinct paths, $\boldsymbol{s}$ and $\hat{\boldsymbol{s}}$;*

(3) a regular point *of system 18 otherwise.*

Figure 1 illustrates these definitions. If z is the state variable, the non-invertible points are all $(w = 0, z \neq 0)$, the singular point $(w = 0, z = 0)$, and the regular points all $(w \neq 0, z)$.

Refine the non-invertible points further:

Definition 6 *A point $\boldsymbol{\sigma}$ is a* truly non-invertible *point of system 18 if it is*

Asymmetric play with control bounds

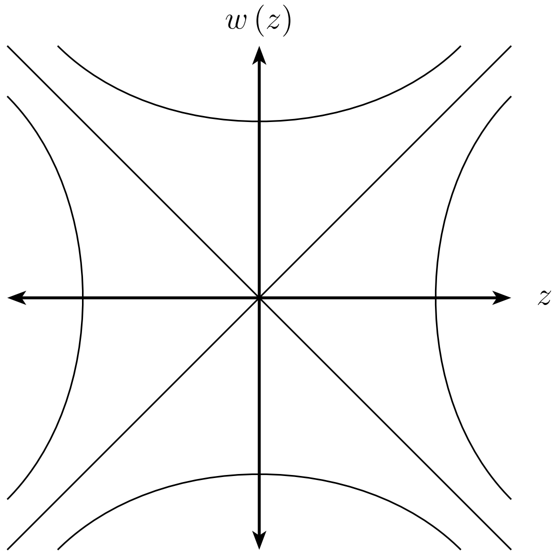

Fig. 1. A simple example of a singularity at the origin: $w\frac{dw}{dz} = z$.

non-invertible and if $\det(\boldsymbol{A}(\boldsymbol{\sigma})) = 0$*. The path on which* $\boldsymbol{\sigma}$ *lies is then truly non-invertible as well.*

Definition 7 *A point* $\boldsymbol{\sigma}$ *is a* quasi-non-invertible *point of system 18 if it is non-invertible but not truly so. The path on which* $\boldsymbol{\sigma}$ *lies is then quasi-non-invertible as well.*

All the non-invertible points in Figure 1 are truly so. Quasi-non-invertible points include kinks, which typically occur during passage between regimes.

Definition 8 *A* candidate MPE strategy *is a path* $\boldsymbol{s}$ *where:*

(1) $\boldsymbol{w} = [s_1, s_2]^\top$ *and* $z = s_3$*;*
(2) $\boldsymbol{w}(z)$ *is a function mapping from* Z *to* $\Re^2$*;*
(3) $\boldsymbol{A}$ *and* $\boldsymbol{f}$ *are defined according to systems 8, 11 and 17 as appropriate.*

3.1 Sufficient conditions to disqualify candidate strategies

Lemma 9 *Candidate MPE strategies containing non-invertible points cannot be considered as MPE strategies.*

This follows directly from the requirement that a candidate strategy be a function defined over the whole domain. Otherwise, other agents cannot otherwise form conjectures about an agent's play under all possible circumstances and, therefore, cannot form a best response.

Lemma 10 *Candidate MPE strategies that set* $W_i(z) > 0$ *for any* $z \in Z$ *cannot be considered as MPE strategies.*

PROOF. As payoff function 2 is bounded above by zero for all z, so is the value function, $V_i\left(\right)$.

For reasons that will become apparent in the numerical analysis, it is also important to eliminate strategy pairs for which $\left(x_1\left(0\right), x_2\left(0\right)\right) = \mathbf{0}$.

Lemma 11 *Strategy pairs which set* $\left(x_1\left(0\right), x_2\left(0\right)\right) = \mathbf{0}$ *cannot be considered as MPE strategies.*

PROOF. Assume that the $x_i^*, i \in \{1, 2\}$ that maximises the RHS of Bellman equation 4 is zero at $z = 0$. The ensuing differential equation is then

$$W_i'\left(z\right) = -\frac{\delta W_i\left(z\right) + \xi^2 + \nu\left(z - \zeta\right)^2}{\beta z}; \tag{19}$$

with solution

$$W_i\left(z\right) = \frac{C_i}{z^{\frac{\delta}{\beta}}} - \frac{\nu}{2\beta + \delta}z^2 + \frac{2\nu\zeta}{\beta + \delta}z - \frac{\xi^2 + \nu\zeta^2}{\delta}; \tag{20}$$

where C_i is a constant of integration.

If $C_i > 0$ then $W_i\left(0\right) = \infty$, violating Lemma 10.

If $C_i < 0$ then $x_i^* = 0$ requires that $W_i'\left(z\right) \leq -2\xi$ which, with equation 19, yields

$$\delta W_i\left(z\right) + \xi^2 + \nu\left(z - \zeta\right)^2 \geq 2\beta\xi z.$$

Replacing the $W_i\left(z\right)$ term with that in equation 20 produces

$$\frac{2\beta\nu}{2\beta + \delta}z^{\frac{2\beta+\delta}{\beta}} + \delta C_i \geq 2\beta\left(\xi + \frac{\nu\zeta}{\beta + \delta}\right)z^{\frac{\beta+\delta}{\beta}}.$$

This inequality fails at $z = 0$ for $C_i < 0$.

If, finally, $C_i = 0$ then differentiating equation 20 with respect to z and requiring that $W_i'\left(z\right) \leq -2\xi$ produces

$$\frac{\left(\beta + \delta\right)\xi + \nu\zeta}{\beta + \delta} \leq \frac{\nu}{2\beta + \delta}z.$$

This also fails at $z = 0$.

3.2 Necessary and sufficient conditions for MPE

Until now, discussion has focussed on disqualifying paths from consideration as MPE strategies. Conditions for determining when candidates are MPE strategies are in Rowat (2002, App. A).

4 Singularities

The unique symmetric linear MPE passes through a singularity in the solution to symmetric equation 10. As an analogous situation may hold in the case of asymmetric play, sufficient conditions for a locus of points through which two paths pass are developed. These conditions are then applied to the systems of differential equations 8 and 11. The first system is found to have a singularity locus based on a conic section, although with points removed. The second does not have a singularity locus. Singularities are not sought in system 17 as its explicit solution may be seen not to yield them.

4.1 Theory

This section uses the following notation and assumptions:

(1) let $\boldsymbol{s} = \boldsymbol{s}(t)$ (thus $\boldsymbol{s} = (w_1(z(t)), w_2(z(t)), z(t), t)$ in the present problem) and rewrite the prototypical differential equation 18 as

$$\boldsymbol{A}(\boldsymbol{s})\,\dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s})\,; \tag{21}$$

where derivatives of $\boldsymbol{s}(t)$ with respect to t are $\ldots \boldsymbol{s}, \ddot{\boldsymbol{s}}, \dddot{\boldsymbol{s}}$ and so on. Therefore the s_n of equation 18 becomes t here. Definition 5, defining non-invertible, singular and regular points, is not modified.

(2) denote singular points by $\boldsymbol{\sigma}$.

(3) set $t = 0$ at $\boldsymbol{\sigma}$; as the system is autonomous in t no generality is lost.

(4) define

$$a_{ij}^{(k)} \equiv \frac{\partial}{\partial s_k} a_{ij}|_{\sigma} \text{ and } f_i^{(k)} \equiv \frac{\partial f_i}{\partial s_k}|_{\sigma};$$

where $[a_{ij}] = \boldsymbol{A}$ and $[f_i] = \boldsymbol{f}$. (As the specific a_{ij} and f_i explored here are members of $\mathcal{C}^{\infty}$ this differentiability assumption is not restrictive.)

(5) when $rank\,(\boldsymbol{A}(\boldsymbol{s})) < n$ let the vector $\boldsymbol{q} \neq \boldsymbol{0}$ (*resp.* $\boldsymbol{c} \neq \boldsymbol{0}$) be a linear combination of the columns (*resp.* columns) of $\boldsymbol{A}$ so that $\boldsymbol{A}(\boldsymbol{s}) \cdot \boldsymbol{q} = \boldsymbol{0}$ (*resp.* $\boldsymbol{c} \cdot \boldsymbol{A}(\boldsymbol{s}) = \boldsymbol{0}$). When $rank\,(\boldsymbol{A}(\boldsymbol{s})) = n - 1$, $\boldsymbol{c}$ and $\boldsymbol{q}$ are unique up to a scalar multiple; when $\boldsymbol{A}$ is symmetric as well let $\boldsymbol{c}' = \boldsymbol{q}$.

Definition 12 *An* m-singularity *is a locus of singular points, each lying on exactly $m \geq 2$ distinct paths.*

Then:

Theorem 13 *Given system 21, in which $a_{ij}, f_i \in \mathcal{C}^1$, sufficient conditions for the point $\boldsymbol{\sigma}$ to be a 2-singularity are:*

(1) (non-invertibility) $\boldsymbol{A}(\boldsymbol{\sigma})$ has rank $n-1$;
(2) (spanning) $\boldsymbol{f}(\boldsymbol{\sigma}) = \boldsymbol{A}(\boldsymbol{\sigma}) \cdot \boldsymbol{r}$ for some n -vector $\boldsymbol{r}$; and
(3) (roots) the quadratic equation

$$\begin{aligned} 0 = \ & \lambda^2 \left[\sum_{i,j,k} a_{ij}^{(k)} c_i q_j q_k\right] \\ & + \lambda \sum_i c_i \left[2 \sum_{j,k} a_{ij}^{(k)} r_j q_k - \sum_k f_i^{(k)} q_k\right] \\ & + \sum_i c_i \left[\sum_{j,k} a_{ij}^{(k)} r_j r_k - \sum_k f_i^{(k)} r_k\right]. \end{aligned} \tag{22}$$

has exactly two distinct real roots in λ given the vector $\boldsymbol{r}$ from the spanning condition.

The intuition behind these conditions is illustrated in Figure 1. The non-invertibility condition ($w = 0$) imposes a barrier, not to paths, but to functions. At the origin, where spanning and non-invertibility hold, however, a slit in the barrier allows crossing paths to remain functions. The roots condition then ensures that the crossing paths are distinct.

The theorem is proven by means of two lemmata.

Lemma 14 *Paths through $\boldsymbol{\sigma}$ that satisfy the conditions of Theorem 13 have one of two slopes.*

PROOF. At $\boldsymbol{\sigma}$ the i^{th} equation of system 21 is

$$\sum_{j=1}^{n} a_{ij} \dot{\sigma}_j = f_i. \tag{23}$$

As $\boldsymbol{A}(\boldsymbol{\sigma})$ is singular, this fails to determine $\dot{\boldsymbol{\sigma}}$. Therefore take advantage of $t_0 = 0$ and expand the elements of equation 23 for

$$\begin{aligned} & \sum_{j=1}^{n} \left\{ a_{ij} + \sum_k a_{ij}^{(k)} \dot{\sigma}_k t + \frac{1}{2} \left[\sum_{k,l} \frac{\partial a_{ij}^{(k)}}{\partial s_l} \dot{\sigma}_k \dot{\sigma}_l + a_{ij}^{(k)} \ddot{\sigma}_k\right] t^2 + \mathcal{O}\left(t^3\right) \right\} \\ & \times \left\{ \dot{\sigma}_j + \ddot{\sigma}_j t + \frac{1}{2} \dddot{\sigma}_j t^2 + \mathcal{O}\left(t^3\right) \right\} \\ & = \left\{ f_i + \sum_{k=1}^{n} f_i^{(k)} \dot{\sigma}_k t + \frac{1}{2} \left[\sum_{k,l} \frac{\partial f_i^{(j)}}{\partial s_k} \dot{\sigma}_k \dot{\sigma}_l + \sum_j f_i^{(j)} \ddot{\sigma}_j\right] t^2 + \mathcal{O}\left(t^3\right) \right\}. \end{aligned} \tag{24}$$

Equality of the coefficients of the powers of t thus produces an infinite number of equations; that in t^0 is simply equation 23. As $rank\left(\boldsymbol{A}\left(\boldsymbol{\sigma}\right)\right) = n-1, \dot{\sigma}_j$ is non-unique. Define it to be

$$\dot{\sigma}_j = r_j + \lambda q_j; \tag{25}$$

where λ is a scalar and q_j is the j^{th} component of $\boldsymbol{q}$. At this point any λ satisfies equation 25.

By equality of the coefficients of the t^1 terms

$$\sum_j a_{ij}\ddot{\sigma}_j = -\sum_{j,k} a_{ij}^{(k)}\dot{\sigma}_j\dot{\sigma}_k + \sum_k f_i^{(k)}\dot{\sigma}_k. \tag{26}$$

As $\boldsymbol{A}\left(\boldsymbol{\sigma}\right)$ is singular, premultiply by $\boldsymbol{c}$ to set the LHS term in equation 26 to zero and substitute in the non-unique $\dot{\sigma}_j = r_j + \lambda q_j$ for

$$\sum_{i,j,k} c_i a_{ij}^{(k)}\left(r_j + \lambda q_j\right)\left(r_k + \lambda q_k\right) = \sum_{i,k} c_i f_i^{(k)}\left(r_k + \lambda q_k\right); \tag{27}$$

a quadratic in λ. As λ's premultipliers are non-singular, this reduces the non-unique $\dot{\sigma}_j$ to no more than two distinct values. The third condition of Theorem 13, on equation 27 (= equation 22), then ensures that $\dot{\sigma}_j$ has two distinct, real values.

As singularity of $\boldsymbol{A}\left(\boldsymbol{\sigma}\right)$ prevented derivation of $\dot{\boldsymbol{\sigma}}$ from equation 23, it also prevents derivation of $\ddot{\boldsymbol{\sigma}}$ from equation 26. The next lemma uses the above technique to derive $\ddot{\boldsymbol{\sigma}}$.

Lemma 15 *A path through a point $\boldsymbol{\sigma}$, satisfying the conditions of Theorem 13, is uniquely identified by its slope at $\boldsymbol{\sigma}$.*

PROOF. From equation 26, singularity of $\boldsymbol{A}\left(\boldsymbol{\sigma}\right)$ allows

$$\ddot{\sigma}_j = p_j + \mu q_j;$$

where μ is a scalar and q_j is again the j^{th} component of $\boldsymbol{q}$. Equating the coefficients of the t^2 terms in equation 24 yields

$$\begin{aligned}\sum_j a_{ij}\dddot{\sigma}_j = {} & \textstyle\sum_{j,k} \frac{\partial f_i^{(j)}}{\partial s_k}\dot{\sigma}_j\dot{\sigma}_k + \sum_j f_i^{(j)}\ddot{\sigma}_j - \sum_{j,k,l}\frac{\partial a_{ij}^{(k)}}{\partial s_l}\dot{\sigma}_j\dot{\sigma}_k\dot{\sigma}_l \\ & \textstyle - \sum_{j,k} a_{ij}^{(k)}\dot{\sigma}_j\ddot{\sigma}_k - 2\sum_{j,k} a_{ij}^{(k)}\dot{\sigma}_k\ddot{\sigma}_j.\end{aligned} \tag{28}$$

Premultiply by $\boldsymbol{c}$ as before for

$$\begin{aligned}0 = {} & \textstyle\sum_{i,j,k} c_i\frac{\partial f_i^{(j)}}{\partial s_k}\dot{\sigma}_j\dot{\sigma}_k + \sum_{i,j} c_i f_i^{(j)}\left(p_j + \mu q_j\right) - \sum_{i,j,k,l} c_i\frac{\partial a_{ij}^{(k)}}{\partial s_l}\dot{\sigma}_j\dot{\sigma}_k\dot{\sigma}_l \\ & \textstyle - \sum_{i,j,k} c_i a_{ij}^{(k)}\left(p_k + \mu q_k\right)\dot{\sigma}_j - 2\sum_{i,j,k} c_i a_{ij}^{(k)}\dot{\sigma}_k\left(p_j + \mu q_j\right).\end{aligned} \tag{29}$$

As this is linear in μ and as μ's premultipliers are non-singular, a unique μ satisfies it. Hence, given $\dot{\sigma}_j$, $\ddot{\sigma}_j$ is unique.

Obtain the coefficients of the higher order terms, $t^n, n > 2$, by further differentiating equation 28. As this shows $\dddot{\sigma_j}$ to be a linear function of the $\ddot{\sigma}$ terms, higher order derivatives, $\frac{d^{n+1}\sigma_j}{dt^{n+1}}$, are also linear in $\frac{d^n\sigma_j}{dt^n}, \forall n > 2$. Thus, given any set of lower order derivatives, $\left\{\dot{\sigma}_j, \ddot{\sigma}_j, \ldots, \frac{d^n\sigma_j}{dt^n}\right\}$, $\frac{d^{n+1}\sigma_j}{dt^{n+1}}$ is unique.

Theorem 13 did not discuss necessary conditions for pragmatic reasons: failures of, for example, its rank condition become quite complicated. Were $rank\left(\boldsymbol{A}\left(\boldsymbol{\sigma}\right)\right) = n-2$, there would be independent n-vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ such that $\boldsymbol{c}_1 \cdot \boldsymbol{A}\left(\boldsymbol{\sigma}\right) = \boldsymbol{c}_2 \cdot \boldsymbol{A}\left(\boldsymbol{\sigma}\right) = \boldsymbol{0}$. Differentiating system 21, produces

$$\boldsymbol{c}_1\left(\boldsymbol{A}'\dot{\boldsymbol{s}} + \boldsymbol{A}\ddot{\boldsymbol{s}}\right) = \boldsymbol{c}_1\boldsymbol{f}'; \text{ and } \boldsymbol{c}_2\left(\boldsymbol{A}'\dot{\boldsymbol{s}} + \boldsymbol{A}\ddot{\boldsymbol{s}}\right) = \boldsymbol{c}_2\boldsymbol{f}';$$

or, by the definition of the $\boldsymbol{c}$ vectors,

$$\boldsymbol{c}_1\boldsymbol{A}'\dot{\boldsymbol{s}} = \boldsymbol{c}_1\boldsymbol{f}'; \text{ and } \boldsymbol{c}_2\boldsymbol{A}'\dot{\boldsymbol{s}} = \boldsymbol{c}_2\boldsymbol{f}'.$$

If

$$\dot{\sigma} = r_j + \lambda_1 q_{1,j} + \lambda_2 q_{2,j};$$

where $\boldsymbol{A}\left(\boldsymbol{\sigma}\right) \cdot \boldsymbol{q}_1 = \boldsymbol{A}\left(\boldsymbol{\sigma}\right) \cdot \boldsymbol{q}_2 = \boldsymbol{0}$ and λ_1 and λ_2 are constants, then each of these defines a conic in λ_1 and λ_2. As both must hold, the conics' intersection defines feasible values of λ_1 and λ_2, some of which may be in the complex hyperplane.

4.2 Singularities in system 8

The preceding analysis of 2-singularities is now applied to system 8 by first examining the points satisfying each of the conditions of Theorem 13 and then assembling them to describe the 2-singularity locus. The 2-singularity locus is a conic section, with some points removed. Some of these removed points are regular rather than singular; others allow no real paths to pass.

4.2.1 The non-invertibility and spanning conditions

Lemma 16 *The points of system 8 satisfying Theorem 13's non-invertibility condition define a cone.*

PROOF. In system 8, Theorem 13's non-invertibility condition becomes

$$g^2 = \frac{1}{4} w_1 w_2. \tag{30}$$

This surface is the union of the generating lines

$$g = \frac{1}{2} p w_1 = \frac{1}{2} \frac{1}{p} w_2; \tag{31}$$

parameterised by the finite $p \neq 0$. The lines in this family are non-parallel and pass through the common $(w_1, w_2, z) = \left(0, 0, \frac{2\xi}{\beta}\right)$.

Lemma 17 *The points of system 8 satisfying Theorem 13's non-invertibility and spanning conditions form a conic section and a line.*

PROOF. As $\boldsymbol{A}(\boldsymbol{\sigma})$ is non-invertible and, by spanning, its columns are proportional to each other, write system 8 as

$$\rho \begin{bmatrix} 2g \\ w_2 \end{bmatrix} = (\beta + \delta) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + 2\nu (z - \zeta) \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

where ρ is some scalar. Solving ρ out of the two equations and applying equation 30 produces

$$(2g - w_2) [g (\beta + \delta) - \nu (z - \zeta)] = 0; \tag{32}$$

thus defining two planes. The first's intersection with equation 30's non-invertibility cone is the degenerate conic $w_1 = w_2 = 2(\beta z - 2\xi)$, a line. The intersection of the second term with the cone is not degenerate; it forms a conic section.

Lemma 18 *Lemma 17's conic section has two branches in* (w_1, w_2, z) *space when*

$$\beta (\beta + \delta) [\beta (\beta + \delta) + 2\nu] > 3\nu^2; \tag{33}$$

and one otherwise.

PROOF. The axis of Lemma 16's cone lies on the $w_1 = w_2$ plane. As the intersection of the cone with Lemma 17's plane is symmetric in w_1 and w_2, whatever branches it has must cross the $w \equiv w_1 = w_2$ plane. Three lines on the symmetric plane are relevant to this proof: the generating lines

$$z = \frac{1}{\beta}\left(\frac{3}{2}w + 2\xi\right) \Leftrightarrow p = -1;$$
$$z = \frac{1}{\beta}\left(\frac{1}{2}w + 2\xi\right) \Leftrightarrow p = 1;$$

and the spanning line when $\boldsymbol{A}$ is non-invertible:

$$z = \frac{w\,(\beta+\delta) + 2\xi\,(\beta+\delta) + \nu\zeta}{\beta\,(\beta+\delta) + \nu}. \tag{34}$$

When the spanning line intersects both generating lines in the same half cone (with vertex at $(w,z) = \left(0, \frac{2\xi}{\beta}\right)$) the conic section has one branch; otherwise it has two. The intersections, at

$$(w,z) = \left(\frac{2\nu\,(\beta\zeta - 2\xi)}{\beta\,(\beta+\delta) + 3\nu}, \frac{2\xi\,(\beta+\delta) + 3\nu\zeta}{\beta\,(\beta+\delta) + 3\nu}\right); \text{ and} \tag{35}$$
$$(w,z) = \left(\frac{-2\nu\,(\beta\zeta - 2\xi)}{\beta\,(\beta+\delta) - \nu}, \frac{2\xi\,(\beta+\delta) - \nu\zeta}{\beta\,(\beta+\delta) - \nu}\right); \tag{36}$$

are both in the same half cone when the product of their w-coordinates is positive:

$$\beta\,(\beta+\delta)\,[\beta\,(\beta+\delta) + 2\nu] < 3\nu^2.$$

Otherwise the intersections are in opposite half cones.

4.2.2 The roots condition

Lemma 19 shows that Theorem 13's roots condition fails in a simple and specific way. As failure occurs on Lemma 17's conic section, the conic's locus of points satisfying Theorem 13's conditions is punctured. The subsequent lemmata then show more general ways in which the roots condition fails.

Lemma 19 *The line defined in Lemma 17 fails to satisfy Theorem 13's condition 3.*

The argument used in the proof is a special case of that in Theorem 13.

PROOF. Along Lemma 17's line, system 8 becomes

$$w_1' + w_2' = \frac{2}{w}\left[(\beta+\delta)\,w + 2\nu\,(z-\zeta)\right]. \tag{37}$$

Substituting this expression into system 8's derivative with respect to z yields

$$\begin{bmatrix} \frac{1}{2}w & \frac{1}{2}w \\ \frac{1}{2}w & \frac{1}{2}w \end{bmatrix} \begin{bmatrix} w_1'' \\ w_2'' \end{bmatrix} + \begin{bmatrix} \delta + 2\nu\frac{z-\zeta}{w} & \frac{1}{2}w_1' \\ \frac{1}{2}w_2' & \delta + 2\nu\frac{z-\zeta}{w} \end{bmatrix} \begin{bmatrix} w_1' \\ w_2' \end{bmatrix}$$
$$= (\beta + \delta) \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} + 2\nu \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The premultiplying vector $[c_1, c_2] = [1, -1]$ cancels the second derivatives to produce

$$\left(2\nu\frac{z-\zeta}{w} - \beta\right)(w_1' - w_2') = 0; \tag{38}$$

the specific form of quadratic condition 22. With the substitution $w_i' = r_i + \lambda q_i$ and $\boldsymbol{q} = \boldsymbol{c}^\top$, equation 38 becomes

$$\left(2\nu\frac{z-\zeta}{w} - \beta\right)(r_1 - r_2 + 2\lambda) = 0;$$

which does not have two real, distinct roots in λ.

The preceding lemma dealt with one of two possible cases of symmetric play satisfying non-invertibility, that corresponding to the generating line with parameter $p = 1$. A similar process confirms that a point on the $p = -1$ line, the other case, does satisfy the conditions of Theorem 13; this produces the 2-singularity at the intersection of the linear solutions to the symmetric differential equation 10, explored more fully in Rowat (2002).

More generally, there are two ways in which the quadratic equation 22 will fail to have distinct, real roots. The next two lemmata examine these possibilities. First, the coefficient of the square term may be zero:

Lemma 20 *In system 8, the coefficient of λ^2 in equation 22 is zero iff $p = 1$.*

PROOF. At non-invertibility, the relationship $g = \frac{1}{2}pw_1 = \frac{1}{2}\frac{1}{p}w_2$ allows system 8 to be written

$$g \begin{bmatrix} 1 & \frac{1}{p} \\ p & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 2 \begin{bmatrix} (\beta + \delta)\frac{1}{p}g + \nu(z - \zeta) \\ (\beta + \delta)pg + \nu(z - \zeta) \end{bmatrix}. \tag{39}$$

Vectors that set $\boldsymbol{A} \cdot \boldsymbol{q} = \boldsymbol{0}$ and $\boldsymbol{c} \cdot \boldsymbol{A} = \boldsymbol{0}$ are

$$\boldsymbol{q} = (1, -p, 0)^\top; \text{ and } \boldsymbol{c} = \left(1, -\frac{1}{p}, 0\right).$$

From equation 22,

$$\sum_{i,j,k} a_{ij}^{(k)} c_i q_j q_k \neq 0;$$

which expands to

$$\begin{aligned}
&\left[\sum_j^2 q_j \left(a_{1j}^{(1)} - p a_{1j}^{(2)}\right)\right] - \frac{1}{p}\left[\sum_j^2 q_j \left(a_{2j}^{(1)} - p a_{2j}^{(2)}\right)\right] \\
&= \left[\left(a_{11}^{(1)} - p a_{11}^{(2)}\right) - p\left(a_{12}^{(1)} - p a_{12}^{(2)}\right)\right] \\
&\quad - \frac{1}{p}\left[\left(a_{21}^{(1)} - p a_{21}^{(2)}\right) - p\left(a_{22}^{(1)} - p a_{22}^{(2)}\right)\right] \\
&= \frac{1}{2} - p - \frac{1}{p}\left[\frac{1}{2}p^2 - p\right] = \frac{3}{2} - \frac{3}{2}p = \frac{3}{2}(1-p) \neq 0.
\end{aligned}$$

The quadratic equation may also fail to have distinct, real roots by having a negative discriminant:

Lemma 21 *Equation 22 has a positive discriminant in system 8 iff*

$$\begin{aligned}
&[\beta+\delta]^2 p^4 - [\beta^2 + 3\nu + \beta\delta] p^3 - [2\beta\delta + \delta^2 - 6\,\nu]\; p^2 \\
&- [\beta^2 + 3\nu + \beta\delta] p + [\beta+\delta]^2 > 0;
\end{aligned} \tag{40}$$

when $p \neq 1$.

PROOF. See Appendix B.

It is not clear how to interpret the failure of condition 40: no real paths pass through the points concerned.

For some calibrations, inequality 40 holds for all p:

Lemma 22 *Inequality 40 holds for all* p *iff*

$$8\nu > 3\beta^2 + 8\beta\delta + 4\delta^2 + \nu\frac{2\beta(\beta+\delta) + 3\nu}{(\beta+\delta)^2}. \tag{41}$$

PROOF. As inequality 40 holds for $p = 0$ only consider those $p \neq 0$. Divide inequality 40 by $p^2 \neq 0$ for

$$(\beta+\delta)^2 q^2 - \left(\beta^2 + 3\nu + \beta\delta\right) q - \left(2\beta\delta + \delta^2 - 6\nu\right) - 2(\beta+\delta)^2 > 0; \tag{42}$$

where $q \equiv p + \frac{1}{p}$. As the coefficient of q^2 is positive its stationary point, at

$$q^* = \frac{\beta(\beta+\delta) + 3\nu}{2(\beta+\delta)^2};$$

is a minimum. Substituting this into inequality 42 yields

$$-\frac{[\beta(\beta+\delta)+3\nu]^2}{4(\beta+\delta)^2} - \left(2\beta\delta + \delta^2 - 6\nu\right) - 2(\beta+\delta)^2 > 0;$$

which may be manipulated to produce condition 41.

4.2.3 The 2-singularity locus

The various statements made above may now be assembled into:

Theorem 23 *The 2-singularity locus of system 8 coincides with the conic section defined in Lemma 17 except when:*

(1) $p = 1$ *(as per Lemma 19); or*
(2) p *is such that inequality 40 fails.*

While the conic section may have one or two branches, the locus never ceases to exist as a result of the auxiliary condition:

Lemma 24 *The auxiliary condition* $w_i \geq -2\xi, i = 1, 2$ *cannot remove the entire 2-singularity locus.*

PROOF. Consider the intersection of the non-invertibility line with parameter $p = -1$ and the objects satisfying non-invertibility and spanning in equation 32; their intersection satisfies the conditions of Theorem 13. This point, identified in equation 35, always satisfies $(w, z) > (-2\xi, 0)$. Therefore, this part of the planar conic is always a 2-singularity.

It may also be of interest to know when the 2-singularity locus has two branches:

Lemma 25 *When condition 41 holds, the 2-singularity locus in system 8 has two branches iff*

$$\frac{\nu(\beta\zeta - 2\xi)}{\beta(\beta+\delta) - \nu} < \xi. \tag{43}$$

and inequality 33 holds.

PROOF. Inequality 33 in Lemma 18 provided a necessary and sufficient condition for the conic to have two branches. The non-invertibility generating line with parameter $p = 1$ intersects the spanning line in equation 34 at $w_1 = w_2 > -2\xi$ iff inequality 43 holds. As, from Lemma 24, the generating line with parameter $p = -1$ always intersects the spanning line at a $w_1 = w_2 > -2\xi$, the proof follows.

As a concluding note, cases in which $rank\left(\boldsymbol{A}\left(\boldsymbol{\sigma}\right)\right) < n - 2$ have not been examined here. Cases of $n - d$ non-invertibility, where $d > 2$, are impossible in system 8 as $n = 2$. The $d = 2$ case requires that $\boldsymbol{A}\left(\boldsymbol{\sigma}\right) = \mathbf{0}$, hence $w_1 = w_2 = g = 0 \Rightarrow z = \frac{2\xi}{\beta}$, conditions that are only satisfied at the apex of the non-invertibility cone. Spanning would then require that $z = \zeta$ so that singularities in this case would require the parameter restriction $2\xi = \beta\zeta$. Given the non-genericity of this restriction, and the costs of presenting the more general theory to address this case, it is not examined here.

4.3 Singularities in system 11

As $w_i \leq -2\xi$ the minimum $rank\left(\boldsymbol{A}\left(\boldsymbol{\sigma}\right)\right)$ in system 11 is $n - 1$. The only singularities that need to be considered here are 2-singularities as the worst case, from the point of view of multiple solutions, sets $rank\left(\boldsymbol{A}\left(\boldsymbol{\sigma}\right)\right) = n - 1$ but Theorem 13's spanning condition holds (so that there is a solution) along with its roots condition (so that there are two solutions).

Theorem 26 *There are no 2-singularities in system 11.*

PROOF. Non-invertibility requires $h = 0$, so that $w_j = 2\left(\beta z - \xi\right)$ and

$$\begin{bmatrix} 0 & 0 \\ \frac{1}{2}w_i & 0 \end{bmatrix} \begin{bmatrix} w_j' \\ w_i' \end{bmatrix} = \begin{bmatrix} \left(\beta + \delta\right) w_j + 2\nu\left(z - \zeta\right) \\ \left(\beta + \delta\right) w_i + 2\nu\left(z - \zeta\right) \end{bmatrix}.$$

If spanning occurs, the first equation produces, with the above implication of $h = 0$ for w_j and z,

$$\left(w_j, z\right) = \left(c, d\right);$$

where c and d are defined in equations 13 and 14. At this point, the second equation must satisfy

$$\frac{1}{2}w_i w_j' = \left(\beta + \delta\right) w_i + 2\nu\left(z - \zeta\right);$$

or, using the first equation,

$$w'_j = 2(\beta + \delta)\left[1 + \frac{1}{w_i}\frac{2\nu(\xi - \beta\zeta)}{\beta(\beta+\delta)+\nu}\right].$$

As $w_i \leq -2\xi$ by construction of the scenario, w'_j is unique.

5 Coding and execution

Traditional finite difference methods are used here, implemented in C for speed. The code is available upon request from the author. The bulk of this section describes the implementation of the conditions for disqualifying candidate paths developed in Section 3. Issues relating to the precision of numerical calculations are also addressed.

5.1 Initial conditions

The finite difference method is implemented by integrating forward in z from a grid of initial conditions, $(w_1(0), w_2(0))$. The upper and lower bounds of the initial conditions grid are somewhat arbitrarily set. The lower bound is usually set at $w_i(0) = -3\xi, i = 1, 2$ as those paths starting from $w_i(0) \leq -2\xi, i = 1, 2$ imply $\boldsymbol{x}(0) = \mathbf{0}$ and are discarded under Lemma 11. Similarly, the upper bound is generally set to $w_i(0) = 0$, implying $x_i(0) = \xi$. This is a reference to the case of symmetric play as $x_i(0) > \xi$ paths there could be discarded for violating a transversality condition (q.v. Rowat, 2002). This upper bound may not be meaningful in the case of asymmetric play.

As unique paths are not generally found by a grid not designed to look for them, the 2-singularity locus of system 8 is also computed. Paths are then computed off of points on this locus. Points on the locus satisfy equations 30 (non-invertibility) and 32 (spanning given non-invertibility). As Lemma 19's distinct roots condition eliminated the possibility that $2g = w_2$, 32 reduces to

$$g = \frac{\nu(z - \zeta)}{\beta + \delta}.$$

With g defined in equation 9, determination of the locus reduces to a problem of three unknowns and three equations (one of them quadratic). The locus is calculated by setting $w_1 = -2\xi$ and using a non-linear solver to determine (w_2, z); w_1 is then varied and the procedure repeated until reaching the singular point on the $w_1 = w_2$ symmetric plane.

5.2 Testing paths

Paths are tested against various conditions as they evolve from initial conditions. Two tests are used to detect truly non-invertible paths. The first tests for a sign change in the determinant. As computation may become slow around these points, the second tests when the determinant falls in absolute value to less than some small tolerance.

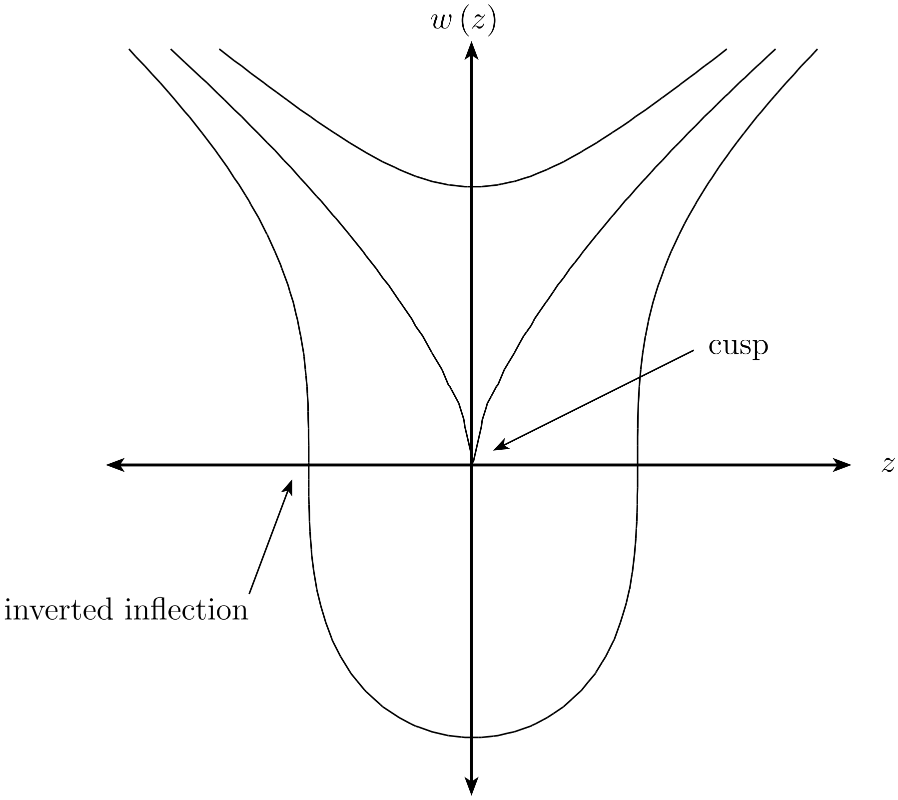

Fig. 2. Small determinants but still functions: $w^2 \frac{dw}{dz} = z$

This second test falsely rejects functions of the sort illustrated in Figure 2. Here, the determinant becomes small as $w \to 0$.

Define points of the sort at $w = 0, z \neq 0$ in Figure 2 by:

Definition 27 *An* inverted inflection point *of a function* $w()$ *is a point* $w(z)$ *at which* $\frac{dw}{dz} = \pm\infty$ *and* $\frac{d^2z}{dw^2} = 0$*. If* $w()$ *is a vector, these relationships must hold for all elements of the vector* $w()$*.*

Therefore:

Theorem 28 *There are no inverted inflection points in system 8.*

PROOF. The definition's first condition holds at system 8's non-invertible points, $g^2 = \frac{1}{4}w_1 w_2$. Substitution into the second condition produces

$$\frac{d^2z}{dw_i^2}|_{NI} = \frac{g - \frac{1}{4}w_j}{(\beta+\delta)\left(g - \frac{1}{2}w_j\right)w_i + 2\nu(z-\zeta)\left(g - \frac{1}{2}w_i\right)}, i \neq j \in \{1,2\}.$$

For this to be zero it must be that $g = \frac{1}{4}w_j$, implying (with the non-invertible condition) that $g = w_i$. Further substitution then allows

$$\frac{d^2z}{dw_i^2}|_{NI} = -\frac{0}{(\beta+\delta)g^2 - \nu(z-\zeta)g}.$$

Necessary and sufficient conditions for $\frac{d^2z}{dw_i^2}|_{NI} = 0$ are now that $g = \frac{1}{4}w_j = w_i \neq 0$ for both $i = 1, 2$, a contradiction.[2]

Theorem 29 *There are no inverted inflection points of system 11.*

PROOF. Now

$$\frac{d^2z}{dw_j^2} = \frac{\frac{1}{2}}{(\beta+\delta)w_j + 2\nu(z-\zeta)} - \frac{h(\beta+\delta)}{[(\beta+\delta)w_j + 2\nu(z-\zeta)]^2};$$

for the non-cornered agent. At non-invertibility $h = 0$ so that

$$\frac{d^2z}{dw_j^2}_{NI} = \frac{\frac{1}{2}}{(\beta+\delta)w_j + 2\nu(z-\zeta)} \neq 0.$$

As there are therefore no invertible inflection points along $w_j()$, there are none in system 11.

Now define points of the sort at $(w, z) = \mathbf{0}$ in Figure 2 by:

Definition 30 *A* cusp *of a function $w()$ is a point $w(z)$ at which*

(1) $w(z)$ is finite; and
(2) $w'(z)$ is not defined; and
(3) either $\lim_{z+} w'(z) = \infty$ and $\lim_{z-} w'(z) = -\infty$, or vice versa.

When $w()$ is a vector, these conditions must hold for all elements of the vector w.

This definition is selected to identify points at which $\det(\boldsymbol{A})$ approaches zero. Notably, it excludes kinks and, by its vertical orientation, many of the usual cusps.

[2] If the denominator is also zero then L'Hôpital's rule sets the numerator to $\frac{1}{2} \neq 0$.

In system 8, the cusp's infinite derivatives require that $g^2 = \frac{1}{4}w_1 w_2$ (non-invertibility). To identify points without a defined derivative, consider a solution path to be the parameterised curve $(w_1(s), w_2(s), z(s))$, with parameter s. As cusps have no tangents, they require $\left(\frac{dw_1(s)}{ds}, \frac{dw_2(s)}{ds}, \frac{dz(s)}{ds}\right) = \mathbf{0}$. Parameterising with $s = t$ reduces this condition to

$$\left(\frac{dw_1}{dz}, \frac{dw_2}{dz}, 1\right)\dot{z} = \mathbf{0};$$

which is equivalent to $\dot{z} = g = 0$.

The code therefore identifies paths for which $g \approx 0$ when approaching $g^2 = \frac{1}{4}w_1 w_2$ in system 8. This will not generally find isolated cusps.

A stronger statement may be made about system 11:

Theorem 31 *There are no cusps in system 11.*

PROOF. For a cusp to exist, the numerator in system 11's first equation

$$w_j' = \frac{(\beta+\delta)\,w_j + 2\nu\,(z-\zeta)}{h};$$

must not change sign as h passes through zero. This requires that

$$z \neq \frac{(\beta+\delta)\,\xi + \nu\zeta}{\beta\,(\beta+\delta)+\nu}.$$

Similarly, as system 11's second equation is

$$w_i' = \frac{[(\beta+\delta)\,w_i + 2\nu\,(z-\zeta)]\,h - \frac{1}{2}w_i\,[(\beta+\delta)\,w_j + 2\nu\,(z-\zeta)]}{h^2};$$

its numerator must change sign while h passes through zero. This requires that

$$z = \frac{(\beta+\delta)\,\xi + \nu\zeta}{\beta\,(\beta+\delta)+\nu};$$

a contradiction.

Tests for other conditions that discard paths from further consideration have also been implemented. Quasi-non-invertible paths are identified by sign tests on the derivatives when moving across $w_i = -2\xi$. Paths setting $W_i(z) > 0$ for some z are detected by means of Bellman equation 4. As this relates $W_i(z)$ and $W_i'(z)$, and as integration determines $W_i'(z)$, $W_i(z)$ is easily calculated.

A calculation implementing Lemma 1 determines when a path along which both agents have cornered stays in the corner.

If any of these conditions is met, or if $\bar{z}$, the (finite) upper limit of integration, is reached, integration terminates and the next path in the grid is selected. An upper limit of integration of $\bar{z} = 1 \times 10^{17}$ has sufficed to ensure that all paths selected by the grid method fail at least one of these conditions before reaching $\bar{z}$. [3]

The integration routines used are from the Numerical Algorithms Group (NAG). [4] For initial value problems with high accuracy requirements, the NAG library recommends Adams methods when the system is not stiff. [5] The present code therefore uses the `d02cjc` ordinary differential equation solver, a variable-order, variable-step Adams method. When `d02cjc` fails to make progress the `d02ejc` ordinary differential equation solver for stiff systems sometimes makes more headway; it uses a variable-order, variable-step backward differentiation formula. NAG sample code often uses the square root of machine zero as the tolerance; on the present hardware this convention implies that `TOL`$= 10^{-8}$.

5.3 Conditioning

As numerical computation uses finite approximations to real numbers, certain operations risk dropping the number of significant digits carried to below acceptable levels. The condition number of a system is a crude approximation to the number of digits lost: when expressed as a power of 10, the exponent reflects the number of significant digits lost.

Condition numbers may be calculated in a number of different ways, usually in agreement as to order of magnitude. Particularly easy to compute is that

[3] The NAG routine `d02cjc` chooses its first step size as a function of $\bar{z}-z$. Increasing the upper limit of integration has caused one or two paths, originally discarded as non-invertible, to become discarded for setting $W_i(z) > 0$, and vice versa. As, in either case, these paths were discarded, this instability is unlikely to affect the equilibrium set.

[4] The implementation code is CLSOL05DA, the Mark 5 C library for Sun SPARC Solaris Double Precision operating systems.

[5] The stiffness ratio of a system is defined as

$$s \equiv \frac{\max_i \mu_i}{\min_i \mu_i}, i \in \{1, 2\};$$

where μ_i is the real component of the i^{th} eigenvalue of the (linearised) system. An ODE system in z is stiff if $\mu_i < 0, i = 1, 2$ and $s >> 1$. A nonlinear system in which s varies is stiff in an interval I when the above hold and $z \in I$ (Itô, 1986, 303.G).

based on the L^{∞} norm:

Definition 32 *For a matrix* $\boldsymbol{A}$ *with elements* a_{ij} *and inverse* $\boldsymbol{A}^{-1}$ *with elements* a_{ij}^{-1} *the* L^{∞} norm condition number *is*

$$cond_{\infty}(\boldsymbol{A}) \equiv \max_{i,j}\{|a_{ij}|\} \times \max_{i,j}\left\{\left|a_{ij}^{-1}\right|\right\}.$$

As a Sun SuperSPARC 1000 with 15 - 16 significant digits is used, convention calls a path *poorly conditioned* when $cond_{\infty} > 10^{10}$ (Judd, 1998, §3.5).

The condition numbers of systems 8 and 11 are

$$cond_{\infty}(\boldsymbol{A}) = \frac{\max\left\{\frac{1}{4}w_1^2, \frac{1}{4}w_2^2, g^2, 1\right\}}{|\det(\boldsymbol{A})|}, w_i \geq -2\xi, i = 1, 2; \tag{44}$$

and

$$cond_{\infty}(\boldsymbol{A}) = \max\left\{\left(\frac{w_i}{2h}\right)^2, 1\right\}, w_i \leq -2\xi, i = 1, 2; \tag{45}$$

respectively. In both cases, a non-invertible $\boldsymbol{A}$ is sufficient for poor conditioning. Call a path *acceptably poorly conditioned* if it sets $\det(\boldsymbol{A}) \approx 0$ and if its neighbours are also poorly conditioned. Discard these paths from consideration as candidates: it is unlikely that the path has been falsely identified as non-invertible (a sufficient condition for discarding it) due to a round-off error as its neighbours suffer the same fate.

Conversely, if a large numerator causes poor conditioning, this may be unacceptable and more careful investigation would be warranted.

6 Results

Table 1 displays the parameter values used in the analyses presented here; the labels indicate whether they give rise to multiple or unique MPE when play is symmetric. Inequality 41, which described when the paths through a singularity would always be real, holds for neither calibration, allowing the possibility of a more complicated 2-singularity locus.

In the 'multiple' calibration, either $(p-1)^2 < 0$ or $p^2 - 161p + 1 > 0$, for inequality 40 to hold. The former yields a contradiction, the latter that $p \lesssim .0062$ or $p \gtrsim 160.9938$ (where the values are each others' reciprocals). Thus, by equation 31, $w_1 \approx 25,919 w_2$ is the most extreme asymmetry possible before the 2-singularity locus disappears. Here, the boundary of $w_i = -2\xi$ is hit when $w_1 = -3$ and $w_2 \approx -1.04$, well within these limits. The 2-singularity locus is therefore not complicated.

	β	δ	ν	ξ	ζ
Multiple symmetric EQ	$\frac{1}{10{,}000}$	0	5.4×10^{-7}	1.5	760
Unique symmetric EQ	$\frac{1}{115}$	$\frac{1}{100}$	5.4×10^{-7}	1.5	760

Table 1
Sets of parameter values most commonly used

The results generated by the multiple symmetric equilibria case are presented in Figure 3 and those of the unique symmetric equilibrium case in Figure 7. The axes represent the initial conditions $w_1(0)$ and $w_2(0)$; $\boldsymbol{w}(0) \in [-3\xi, -2\xi]$ corresponds to starting play in the corner, $\boldsymbol{x}(0) = \mathbf{0}$. The regions displayed represent the outcome of the paths starting with these values. Under both calibrations the linear equilibria are unique in the class of linear equilibria, consistent with the result in Lockwood (1996).

6.1 Multiple symmetric equilibria

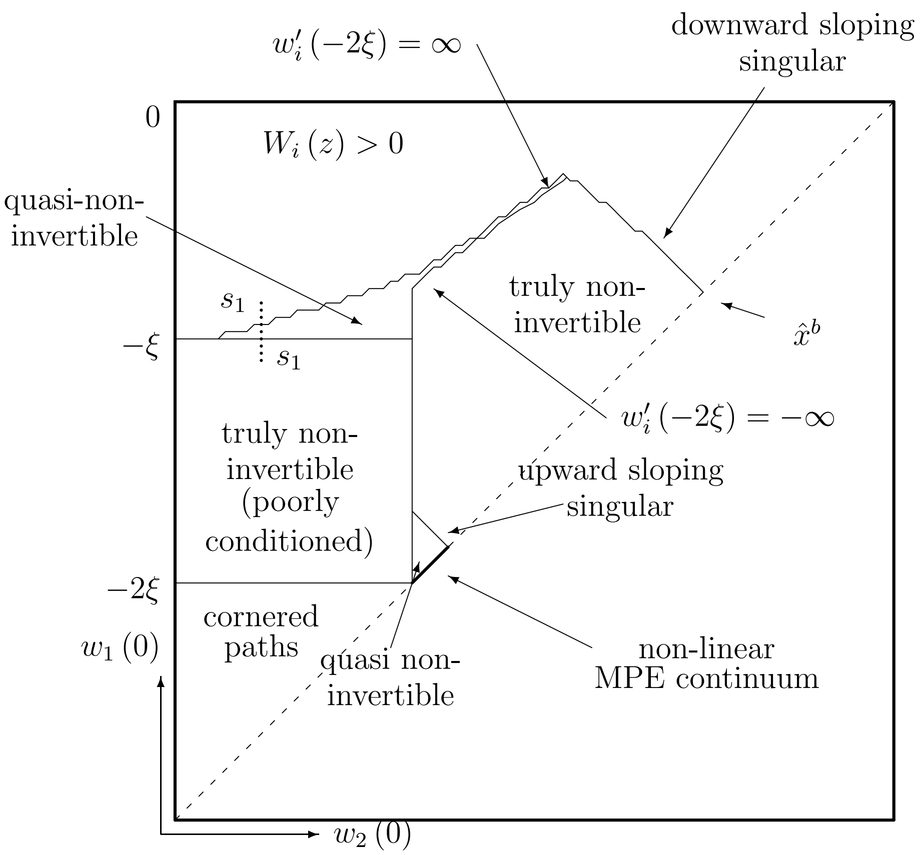

Fig. 3. Outcome as a function of 100 × 100 initial conditions (multiple symmetric equilibrium)

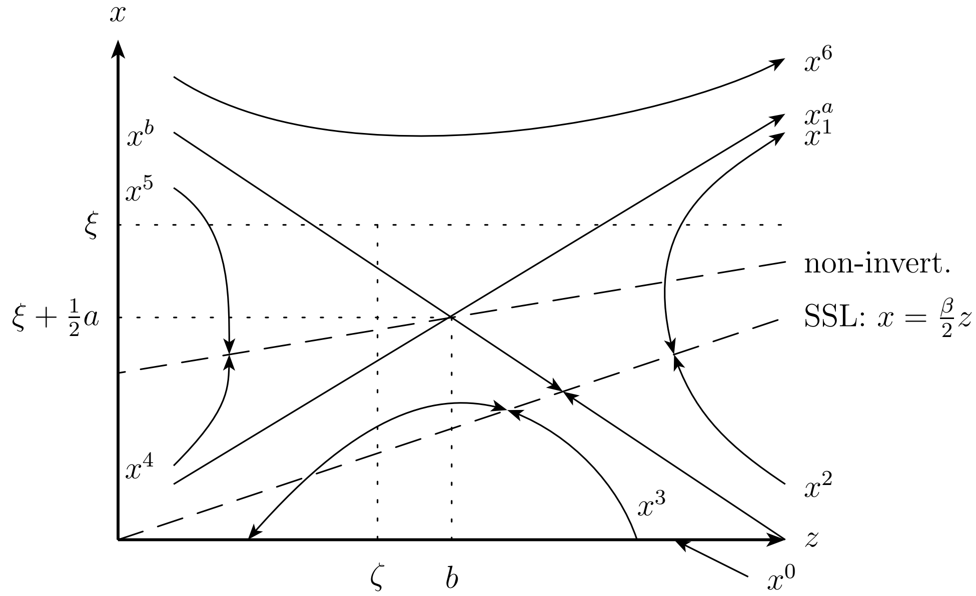

Fig. 4. Symmetric play results from Rowat (2002, Figure 2)

The parameter values generating multiple symmetric equilibria produce Figure 3. Comparing the results along the symmetric axis to those in Figure 4, reproduced from Rowat (2002), finds that they match. Paths with $w_i(0) < -2\xi$ never set $x_i > 0$, corresponding to the x^0 analytical paths. Above $w_i(0) = -2\xi$ there is a region of MPE paths, the $\hat{x}^3$ family. The subsequent truly non-invertible paths correspond to the x^4 and x^5 analytical paths. Finally, those paths setting $W_i(z) > 0$ are the unbounded x^6 paths.

Results from asymmetric play reveal no evidence of regions of asymmetric MPE. Assymetries as small as machine zero lead the asymmetric neighbours of the continuum of MPE paths to quasi-non-invertibility. This occurs as the symmetric MPE strategies obey equations 17 after cornering; their neighbours obey equations 11.

As the grid applied does not reveal asymmetric MPE, one dimensional or isolated MPE are now sought in the boundaries between regions.

First examine paths through the 2-singularity locus. Because normal integration is not possible near this locus the following method is used:

(1) integration off of the locus initially occurs by solving quadratic equation 27 for λ. Knowledge of $\boldsymbol{r}$ then determines the two slopes, allowing initial movement off the locus to be calculated.
(2) when the absolute value of $det(\boldsymbol{A})$ exceeds zero by some tolerance, the NAG integration routines are again used.

Figure 5 displays the results of this procedure. The paths increasing to the left are $\hat{x}^b$ and its asymmetric siblings; those increasing to the right are x^a and its siblings. The two loci of points for which these paths intersect $z = 0$ produce

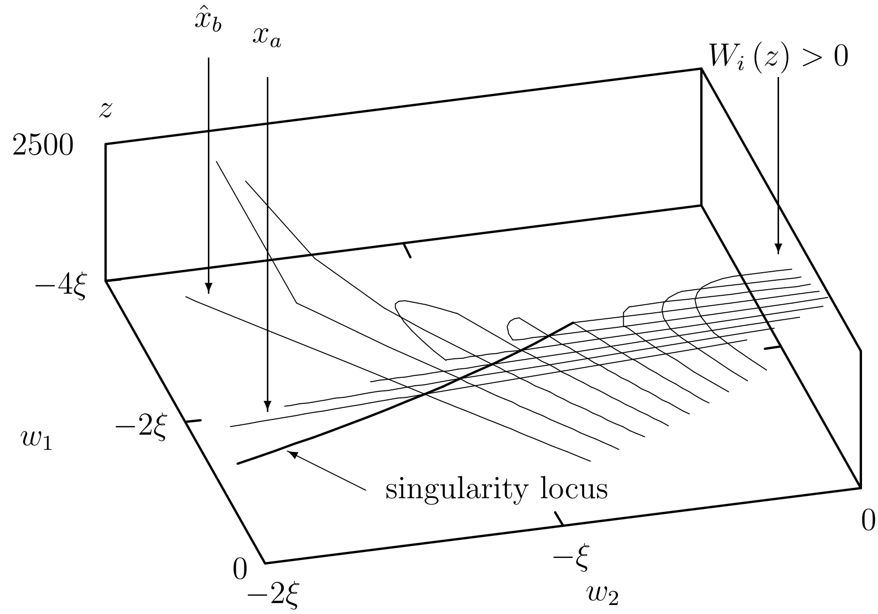

Fig. 5. Paths through and near the 2-singularity locus

the diagonal boundaries off the symmetric axis in Figure 3.

None of the asymmetric paths explored are found to be equilibria. The increasing paths set $W_i(z) > 0$ while the decreasing paths become quasi-non-invertible when they intersect $w_i = -2\xi$. In the latter cases, paths closer to the far end of the singularity locus actually loop back on themselves, passing through the same point on the singularity locus. There is therefore no evidence of new MPE paths along the singular locus.

Now investigate the boundary between the cornered and the (poorly conditioned) truly non-invertible paths simply divides paths into those with zero emissions initially and those with positive emissions.

The next boundaries to its north (near $w_1(0) = -\xi$) are explored by integrating along a section of initial conditions, $s_1 - s_1$, as displayed in Figure 6. The southernmost of these paths becomes non-invertible with w_2 very negative and large. The second path starts at $w_1(0) > -\xi$; this returns from the corner, but becomes quasi-non-invertible in doing so. The third path returns properly from the corner but ultimately sets $W_i > 0$.

This pattern is repeated to the east-north-east, on the other side of $w_2(0) = -2\xi$, with a single difference: while still becoming truly non-invertible, the southernmost paths now start in the interior; they do not corner and they remain well conditioned. The quasi-non-invertible paths again fail to leave the

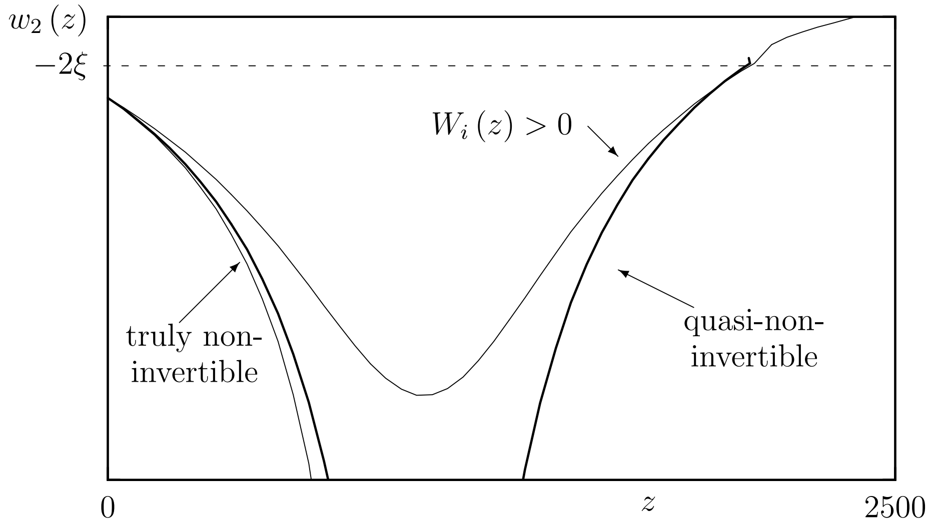

Fig. 6. Projections of paths through the $s_1 - s_1$ section of initial conditions

corner, while those that set $W_i(z) > 0$ do leave the corner (or never corner if the initial conditions are large enough).

Integration along paths close to the NW border between the $W_i(z) > 0$ paths and the quasi-non-invertible paths to their south shows that this border sets $w'_i(-2\xi) = \infty$. Similarly, the border to its south, separating quasi- and truly non-invertible paths marks $w'_i(-2\xi) = -\infty$. As none of these boundaries allow the paths through them to be MPE, no evidence is found for asymmetric MPE.

Thus, confidence that asymmetric MPE strategies do not exist is high. None are found, either by the initial grid search or by more specific searches attuned to one dimensional loci; all of the poorly conditioned paths are acceptably so; there is no evidence of cusps.

6.2 Unique symmetric equilibrium

As the results here are simpler than those above, they are presented in more cursory fashion. Again, there is no evidence of asymmetric MPE. The symmetric axis of Figure 7 is consistent with analytical results under symmetric play. Initially paths corner; now these paths are not just the x^0 family but $\hat{x}^a$ and some x^4 paths. When starting values are large enough for paths not to corner they are the x^4 and x^5 paths. The grid of initial conditions displayed in Figure 7 does not extend to sufficiently high initial conditions to demonstrate the $\hat{x}^b$ path and the x^6 family but individual integration with initial conditions up to $w_i(0) = \xi$ do reveal them.

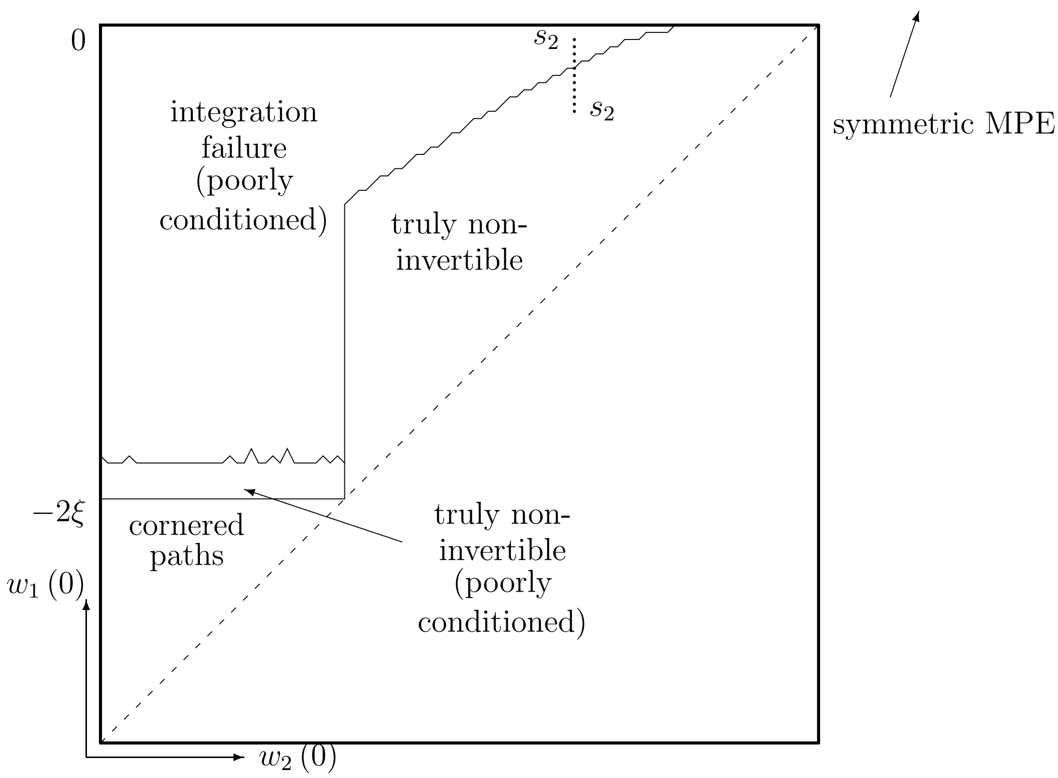

Fig. 7. Outcome as a function of 100 × 100 initial conditions (unique symmetric equilibria)

Novel here are the large areas in which even the hardier `d02ejc` integration routine fails. All individual paths explored approach the non-invertible $h = 0$, accompanied by an exponential decline in one of the $w_i(z)$; these are therefore acceptable examples of poor conditioning. The border through which section $s_2 - s_2$ (indicated in Figure 7) passes illustrates: to its north, agent 2 corners to approach system 11's non-invertible surface; to its south, agent 2 hits the system 8 non-invertible surface before cornering. Again, cusps have not been detected.

7 Discussion

This paper suggests that the non-linear equilibria of Rowat (2002) are not robust. Furthermore, it finds no evidence of new asymmetric MPE. This is a weaker statement than a formal proof, which might be constructed by extending the proof technique used in Rowat (2002). In that case, all possible families of solutions to equation 10 were identified on a phase diagram. Asymmetric play simply adds dimensions: families of paths could be identified in $\Re^3$ and analogous arguments used. For example, if the paths to the 'right' of the singularity locus in Figure 5, between the paths through the singularity, converged to x^a they could be dismissed.

The symmetric system has one fewer equation and one fewer unknown than does that presented here. As the equations are not linear, this counting technique should not account for the non-robustness of the continuum. Rather it seems that symmetric play is simply a special case, exhibiting non-generic properties. This feature of symmetric play may also produce the continuum results found in non-linear quadratic models (Dockner and Sorger (q.v. 1996) and the examples in Rowat (2002)).

Initially Wirl and Dockner's model of a monopoly supplier and a monopsony demander (Wirl and Dockner, 1995) seems to provide a counter-example to this possible link between agent symmetry and multiplicity. In spite of their agents' asymmetry, they too find a continuum of MPE. They do this analytically, first summing the two differential equations describing candidate value functions into a single one to define a new candidate value function. They then follow the approach of Tsutsui and Mino (1990). This second step reduces confidence in their analysis for the reasons outlined in Rowat (2002).

The pleasant computational implication of the suspected uniqueness of MPE strategies is that calculation of MPE for more than two agents is simplified. Without control bounds this reduces to the solution of coupled Riccati equations. Control bounds likely complicate calculations. Uniqueness may also have practical consequences for those situations reflected in this model: Pareto improvements cannot be obtained by coordinating on superior Nash equilibria (cf. Radner, 1998, p. 8).

Finally, while the control bounds considered here are one-sided, analysis could be extended easily to include an upper bound as well.

A The linear quadratic model with asymmetric agents

Generalise the instantaneous utility functions 2 to

$$u_i\left(x_i, z\right) = -\left(x_i - \xi_i\right)^2 - \nu_i\left(z - \zeta_i\right)^2, i = 1, 2;$$

but retain the linear equation of motion 1. System of equations 8 therefore becomes

$$\begin{bmatrix} g & \frac{1}{2}w_1 \\ \frac{1}{2}w_2 & g \end{bmatrix} \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} \left(\beta + \delta_1\right) w_1 + 2\nu_1\left(z - \zeta_1\right) \\ \left(\beta + \delta_2\right) w_2 + 2\nu_2\left(z - \zeta_2\right) \end{bmatrix}; \qquad \text{(A.1)}$$

where

$$g \equiv \frac{1}{2}(w_1 + w_2) - \beta z + (\xi_1 + \xi_2).$$

The non-invertibility locus is still defined by $g^2 = \frac{1}{4} w_1 w_2$ but the spanning condition is now

$$(\beta + \delta_1) w_1 w_2 - (\beta + \delta_2)(2g) w_2 + 2\nu_1 (z - \zeta_1) w_2 - 2\nu_2 (z - \zeta_2)(2g) = 0;$$

an algebraic variety of degree two. This reduces to equation 32 if agents are symmetric. Sample vectors involved in non-invertibility and spanning, according to the notation of Definition 12 and Theorem 13, are then

$$\begin{aligned}
\boldsymbol{q} &= \left[-sign(w_i)\sqrt{\frac{w_1}{w_2}}, 1, 0\right]^\top; \\
\boldsymbol{c} &= \left[-sign(w_i)\sqrt{\frac{w_2}{w_1}}, 1, 0\right]; \text{ and} \\
\boldsymbol{r} &= g \times \left[2(\beta + \delta_2), 2\frac{\nu_2}{g}(z - \zeta_2), 1\right]^\top.
\end{aligned}$$

System of equations 11 becomes

$$\begin{bmatrix} h & 0 \\ \frac{1}{2} w_i & h \end{bmatrix} \begin{bmatrix} w_j' \\ w_i' \end{bmatrix} = \begin{bmatrix} (\beta + \delta_j) w_j + 2\nu_j (z - \zeta_j) \\ (\beta + \delta_i) w_i + 2\nu_i (z - \zeta_i) \end{bmatrix};$$

where equation 12 is now

$$h \equiv \xi_j + \frac{1}{2} w_j - \beta z.$$

Finally, equation 17 becomes

$$w_i(z) = K_i z^{\frac{\beta + \delta_i}{\beta}} + 2\nu_i \left(\frac{\zeta_i}{\beta + \delta_i} - \frac{z}{\delta_i}\right) = z\left\{K_i z^{\frac{\delta_i}{\beta}} - \frac{2\nu_i}{\delta_i}\right\} + \frac{2\nu_i \zeta_i}{\beta + \delta_i};$$

where K_i is a constant of integration.

Numerical integration is performed using the calibration in Table A.1.

The results are presented in Figure A.1. The labelling is slightly simplified here: although many starting values lead to integration failure regions are still labelled according to logic presented below. The 2-singularity locus is not calculated for this model. There is no evidence of regions of MPE.

Asymmetric play with control bounds

	β	δ	ν	ξ	ζ
Agent 1	$\frac{1}{10{,}000}$	0	5.4×10^{-7}	1.5	760
Agent 2		$\frac{1}{100}$	5.4×10^{-7}	1.5	760

Table A.1
Test asymmetric parameter values

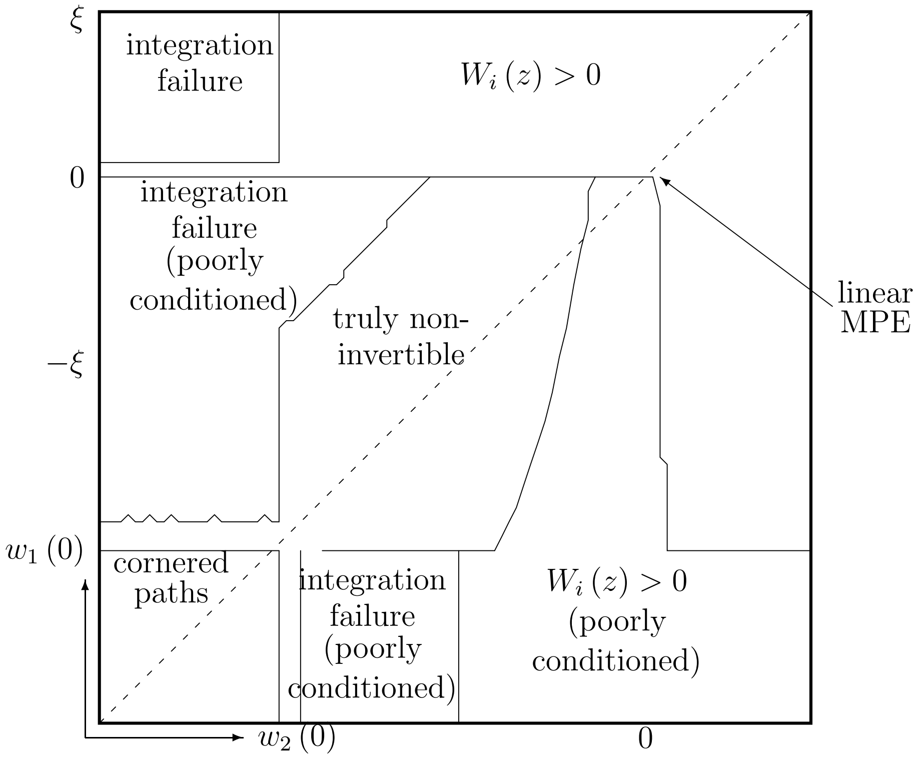

Fig. A.1. Outcome as a function of 100×100 initial conditions (asymmetric players)

The multiple symmetric equilibria of Figure 3 may be regarded as a limit of the results in Figure A.1 as $\delta_2 \to 0$. To approximate this, a series of $\delta_2 = \left\{\frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \frac{1}{1000000}\right\}$ has been explored. As expected, these calibrations increasingly resemble Figure 3. The poorly conditioned zone setting $W_i(z) > 0$ is replaced by a well conditioned one; the region of failed integration in the north-west disappears and that below it shrinks, its northern boundary becoming a region of quasi-non-invertibility. This sense of continuity allows the guess that these regions of integration failure might be truly non-invertible, by analogy to Figure 3.

B Simplifying the discriminant

This appendix proves Lemma 21. A quadratic expression in r_1 as a function of $(p, \beta, \delta, \nu, \xi, \zeta)$ is first produced; the expression for the discriminant follows.

As the general quadratic equation 22 has two roots, let $w_i' = r_i, i = 1, 2$, so that r_i does not need correction by a λ term to produce a slope. Note also that

$$r_3 = g = \frac{2\xi - \beta z}{1 - p - \frac{1}{p}};$$

where the last equality comes from the definition of g and the generating lines of equation 31. When $g \neq 0$, write system 39 (8 at non-invertibility) as

$$\left(r_1 + \frac{1}{p} r_2\right) \begin{bmatrix} 1 \\ p \end{bmatrix} = 2(\beta + \delta) \begin{bmatrix} \frac{1}{p} \\ p \end{bmatrix} + 2\nu (z - \zeta) \frac{\left(1 - p - \frac{1}{p}\right)}{(2\xi - \beta z)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{B.1}$$

Divide the second by $p \neq 0$ to equate their RHS for

$$(\beta + \delta) = \nu (z - \zeta) \frac{\left(1 - p - \frac{1}{p}\right)}{(2\xi - \beta z)};$$

as $p \neq 1$. When $z \neq \frac{2\xi}{\beta}$,

$$z = \frac{2\xi (\beta + \delta) + \left(1 - p - \frac{1}{p}\right) \nu \zeta}{\beta (\beta + \delta) + \left(1 - p - \frac{1}{p}\right) \nu}.$$

Violating the assumed $z \neq \frac{2\xi}{\beta}$ causes $g = 0$ and $rank\left(\boldsymbol{A}(\boldsymbol{\sigma})\right) = 0$, which violates Theorem 13's first condition. Substituting the expression for z into equation B.1 produces

$$r_2 = 2(\beta + \delta)(1 + p) - p r_1. \tag{B.2}$$

Differentiate system 8 at non-invertibility and premultiply by $\begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{p} \end{bmatrix}$ to produce

$$\begin{bmatrix} g' - \frac{1}{p}\frac{1}{2} r_2 & \frac{1}{2} r_1 - \frac{1}{p} g' \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = (\beta + \delta) \left(r_1 - \frac{1}{p} r_2\right) + 2\nu \left(1 - \frac{1}{p}\right).$$

As $g' = \frac{1}{2}(r_1 + r_2) - \beta$ this simplifies to

$$\begin{aligned} &\tfrac{1}{2} r_1^2 - \tfrac{1}{p}\tfrac{1}{2} r_2^2 + \left(1 - \tfrac{1}{p}\right) r_1 r_2 \\ &= (2\beta + \delta) r_1 - \tfrac{1}{p} (2\beta + \delta) r_2 + 2\nu \left(1 - \tfrac{1}{p}\right). \end{aligned} \tag{B.3}$$

Substitute equation B.2 into equation B.3 for

$$0 = r_1^2 + \frac{4}{3}\frac{\left[(\beta+\delta)(1+p)\left(2-\frac{1}{p}\right)-(2\beta+\delta)\right]}{(1-p)} r_1$$
$$\frac{4}{3}\frac{\left[(\beta+\delta)\left(\frac{1}{p}+1\right)(2\beta+\delta)-(\beta+\delta)^2\left(\frac{1}{p}+2+p\right)-\nu\left(1-\frac{1}{p}\right)\right]}{(1-p)};$$

whose discriminant is

$$\frac{16}{9p^2(p-1)^2}\left\{[\beta+\delta]^2 p^4 - \left[\beta^2+3\nu+\beta\delta\right]p^3 - \left[2\beta\delta+\delta^2-6\nu\right]p^2 - \left[\beta^2+3\nu+\beta\delta\right]p + [\beta+\delta]^2\right\}.$$

When $p \neq 1$, a positive discriminant therefore requires that

$$[\beta+\delta]^2 p^4 - [\beta^2+3\nu+\beta\delta]p^3 - [2\beta\delta+\delta^2-6\nu]p^2 - [\beta^2+3\nu+\beta\delta]p + [\beta+\delta]^2 > 0.$$

References

Dockner, E., Sorger, G., 1996. Existence and Properties of Equilibria for a Dynamic Game on Productive Assets. Journal of Economic Theory 71, 209–27.

Itô, K. (Ed.), 1986. Encyclopedic Dictionary of Mathematics, 2nd Edition. MIT Press.

Judd, K. L., 1998. Numerical Methods in Economics. MIT Press.

Lockwood, B., 1996. Uniqueness of Markov-perfect equilbrium in infinite-time affine-quadratic differential games. Journal of Economic Dynamics and Control 20, 751–65.

Radner, R., March 1998. Notes on Implementing Sustainable Development, background material for 8 April, 1998 presentation.

Rowat, C., 2002. Non-linear strategies in a linear quadratic differential game with bounded controls. Working Paper 02-11, University of Birmingham, Department of Economics.

Tsutsui, S., Mino, K., 1990. Nonlinear strategies in Dynamic Duopolistic Competition with Sticky Prices. Journal of Economic Theory 52, 136–61.

Wirl, F., Dockner, E., 1995. Leviathan governments and carbon taxes: Costs and potential benefits. European Economic Review 39 (6), 1215–1236.