Functional Nash equilibria in commons games

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Abstract

This paper explores functional Nash equilibria in three static commons problems. The first yields a non-existence result. Two linear equilibrium strategies are found in the second. Unlike the result in Klemperer and Meyer (1989, Eca), this is unaffected by the domain of the stochastic variable. The third model finds two FNE when the second's strategy space is expanded to allow transfers. While these equilibria are improvements over their equivalents in the second model, the models cannot be Pareto ranked.

Key words: Functional Nash equilibria, transfers, side payments, commons problem, uncertainty

1 Introduction

This paper explores transfers in a commons game. Practically, transfers are the carrots of international diplomacy. While much of it involves in kind reciprocation, monetary transfers are also common. A famous example is the elevation of Israel and Egypt to the first and second largest recipients of US aid after President Carter's Camp David accords. Tied aid, whereby a donor requires that a fraction of the aid be used to purchase its goods or services, is standard.

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In the context of commons problems, transfers play a role in the political discussion on coordinating greenhouse gas emissions. The Kyoto Protocol provides for international 'trade' in emissions in order to meet emissions reduction targets. This may be interpreted as a conditional transfer mechanism: one country provides a financial transfer to another in return for an agreement on emissions from the latter. It is this situation which motivates the examples presented here.

Pigouvian taxes, Coasian assignment of property rights, and Groves-Clark-Ledyard mechanisms, the standard approaches to commons problems, all depend on the presence of a social planner. As this requirement is not satisfied in the context of international climate change negotiations, it is avoided here. Instead, agents are given a two dimensional strategy space, allowing them to offer transfers contingent upon behaviour as reward schemes. [2] The central question to be explored, therefore, is that of whether self-enforcing voluntary transfers offer the possibility of Pareto improvement.

To analyse transfers, this paper specifies strategies as functions. To understand why, consider the usual case, in which agents select single values of play. As other agents' play is held fixed in equilibrium, voluntary transfers incur a cost without offering the possibility of altering play. Unless the game is specified in such a way as to yield other benefits to these transfers, agents will not make them in equilibrium. Consideration of transfers therefore requires a solution concept that allows agents to anticipate the response of others to their transfers.

Klemperer and Meyer (1989) developed such a solution concept, the functional Nash equilibrium. There, duopolists faced demand shocks after they choose their strategies. They therefore selected supply functions to respond flexibly to the uncertainty. No commitment technology was required as the functions were chosen so that equilibrium outcomes were *ex post* optimal.

The motive for considering strategy functions was perhaps even stronger in Green and Newbery (1992), which studied competition in the British electricity generation industry, as British firms are legally required to submit supply functions the day before generation. A real auctioneer then equated supply and demand. While this paper followed Klemperer and Meyer (1989) closely, it made demand a function of price and time, instead of Klemperer and Meyer's price and a shock.

This institutional motive for using functions returns to the origins of this literature. Grossman (1981) noted that Cournot and Bertrand conjectures implicitly gave incumbent firms and potential entrants asymmetric powers of con-

[2] Some label games with these larger strategy spaces 'linked games' (q.v. Carraro, 1997).

tracting with potential customers. Allowing firms to specify supply functions (interpreted as binding contracts) provided a framework in which incumbents and entrants could be treated symmetrically.

Grossman (1981) found a multiplicity of supply function equilibria, which Klemperer and Meyer (1989) recognised as reducible by the introduction of uncertainty. To illustrate, consider duopolists setting production levels of a homogeneous good. As the firms have preferences over both production decisions, almost all output combinations may be described as the intersection of their indifference curves. This generates two sources of multiplicity. First, given any output pair at which indifference curves intersect, each firm can choose a supply function tangent to the other's indifference curve. Second, as the functions are only constrained by the equilibrium tangency requirements and weak conditions to ensure uniqueness, a multiplicity of functions can support an equilibrium at any output pair if non-linear strategies are permitted.

The introduction of demand uncertainty in Klemperer and Meyer (1989) addressed this second source of multiplicity. Now supply functions must support equilibria in each of the economies indexed by the demand shock. As the support of the stochastic term increases, so does the range over which optimality must hold. This serves to refine the set of functions supporting equilibria. Robson (1981) found a similar result in a much simpler model. There, the introduction of uncertainty reduces the equilibrium set to a singleton, containing only the Bertrand outcome.

The bulk of this paper analyses models without transfers in order to develop a benchmark against which to compare the transfers. The main model seems more similar to Robson (1981) than it is to Klemperer and Meyer (1989): the introduction of any uncertainty performs all the refinement.

Modelling of non-cooperative transfers raises a question of how the transfer takes place: does its recipient act first and hope that the transfer is then given or does the donor make the transfer and hope that it is understood as a function rather than a lump sum? A standard means of addressing this is to model the problem as a repeated game. Alternatively, Nash refinements such as the "coalition proof" equilibrium of Bernheim, Peleg, and Whinston (1987) provide a *via negativa* to the question, eliminating equilibria that may be undone by self-enforcing deviations by coalitions. Neither of these are appropriate to the present context, which presents two player static games for the sake of tractability.

Thus, the problem is solved here by assumption: agents offer functions; there is no enforcement problem. This may be plausible even in the international realm, as nations may bind themselves to international norms by national law. European Union member adopt national implementation legislation facilitating enactment of EU decisions; the US constitution proclaims that "Treaties made, or which shall be made, under the Authority of the United States, shall be the supreme Law of the Land".

This approach is therefore similar to that in the common agency analysis of Bernheim and Whinston (1986). There, principals offered outcome contingent reward schedules to an agent. The rewards were paid by assumption, and could be negative. The agent took an unobservable action which determined a probability distribution over a finite set of outcomes. This structure meant that principals could be regarded as setting their rewards in two stages, the first 'cancelling' out the other principals' rewards and the second imposing their own. In doing so, they aligned their incentives by implicit transfers, mediated through rewards to the agent. This allowed a strong result: equilibria are efficient in the sense of implementing the action at minimum cost. Under a number of special but appealing conditions, the fully co-operative outcome could also be implemented.

In the present analysis, schedules are also selected, with players also imagining themselves to choose an aggregate variable directly. (Thus, analysis also resembles the Groves and Ledyard (1977) approach to the optimal allocation of public goods.) Here, though, there is no agent, removing the requirement that its participation constraint be satisfied. Thus, the variable that it set in Bernheim and Whinston (1986) remains under the control of players here. This may account for the present inefficiency results. Finally, in the model with transfers, preferences are already congruent over values of the state variable and transfers are bounded below by zero.

Section 2 presents the functional Nash equilibrium concept and related terminology. Section 3 presents a non-existence result in a simple model without transfers. Section 4 presents an existence result in a quadratic model; the model's Pareto frontier is developed in Section 5. Section 6 extends the quadratic model to include transfers. Section 7 concludes.

2 Functional Nash equilibria

In the class of games examined below a state variable is defined as follows:

$$z \equiv x(z) + y(z) + \varepsilon; \tag{1}$$

where agent 1 chooses the function $x(z)$, agent 2 chooses $y(z)$ and ε is a random variable with support $[-\bar{\varepsilon}, \bar{\varepsilon}]$. It is assumed that $\hat{\varepsilon} \neq 0$ unless otherwise stated. Let $f(\varepsilon)$ be the density function of ε and $E[\varepsilon] = 0$.[3]

The state variable, $z(\varepsilon)$, is a function whose domain is the support of ε. The reasons for this are as follows. First, the problem is otherwise unintelligible: if an ε induces no z, no $x(z)$ or $y(z)$ can be derived either. Second, if z were multivalued for some realisations of ε, the equivalence between expected utility and pointwise maximisation which is exploited below fails. This, in turn, prevents the *ex post* optimality of the equilibrium functions. Requiring that z be a function is *ad hoc*, but standard.[4]

Strategies x and y must also be defined over the whole domain, $z \in \Re^*$, the extended reals. Were they not, agents would be unable to form complete conjectures of others' play, and therefore not be able to select their own best responses. Therefore:

Definition 1 *A pair of strategy functions $(x(z), y(z))$ is* admissible *if $x(z)$ and $y(z)$ are defined over $z \in \Re^*$ and if the mapping that they, through equation 1, induce between ε and z is a function with domain $[-\bar{\varepsilon}, \bar{\varepsilon}]$.*

A consequence of admissibility is that:

Lemma 2 *An admissible function pair $(x(z), y(z))$ produces, through equation 1, an injective $z(\varepsilon)$.*

PROOF. Assume that ε_1 and ε_2 both induce the same z. The left hand side and first two right hand side terms of equation 1 are the same for both ε_1 and ε_2 but the last term is different. This cannot be. $\square$

Define expectations over realisations of ε. Therefore, an admissible strategy pair $(x(z(\varepsilon)), y(z(\varepsilon)))$ yields the pair of expected payoffs

$$(E[u(x(z(\varepsilon)))|y(z(\varepsilon))], E[v(y(z(\varepsilon)))|x(z(\varepsilon))]);$$

where $u(x(z))$ and $v(y(z))$ are the objective functions of agents 1 and 2. Now, for the two agent game, the equilibrium concept is defined as follows:

[3] The restriction on the first moment plays no role beyond determining expected utilities.

[4] Klemperer and Meyer (1989, p. 1250) assume that, "if a market-clearing price does not exist, or is not unique, then firms earn zero profits... [T]his assumption ensures that such an outcome will not arise in equilibrium, but the assumption is not an important constraint on firms' behavior". While Grossman's first two theorems allow multiple prices to equate supply and demand, he works with a price function (Grossman, 1981, p. 1157).

Definition 3 *If:*

(1) $(x^*(z(\varepsilon)), y^*(z(\varepsilon)))$ *is admissible; and*
(2) $E[u(x^*(z(\varepsilon)))|y^*(z(\varepsilon))] \geq E[u(x(z(\varepsilon)))|y^*(z(\varepsilon))]$; *and*
(3) $E[v(y^*(z(\varepsilon)))|x^*(z(\varepsilon))] \geq E[v(y(z(\varepsilon)))|x^*(z(\varepsilon))]$ *for all admissible* $(x(z(\varepsilon)), y^*(z(\varepsilon)))$ *and* $(x^*(z(\varepsilon)), y(z(\varepsilon)))$

then $(x^*(z(\varepsilon)), y^*(z(\varepsilon)))$ *forms a* functional Nash equilibrium (FNE).

Finally,

Definition 4 *The* supportable domain *of a game is the set of* z *that may be supported in a FNE of that game.*

The following sections present a non-existence and an existence result, respectively.

3 Non-existence: payoffs linear in control

A non-existence result is presented here. It reflects an agent's attempt to set its control to a constant, something that the stochastic term prevents.

Problem 5 (linear) *Agents 1 and 2 have objective functions*

$$\begin{aligned} u &= -x - (z-\delta)^2; \text{ and} \\ v &= -y - (z+\delta)^2; \end{aligned}$$

respectively, where $\delta \in \Re_+$ *is a parameter. Agent 1 solves*

$$\max_{x(z)} E\left[-x - (z-\delta)^2\right]; \; z(\varepsilon) = x(z) + y^*(z) + \varepsilon$$

where $y^*(z)$ *is some fixed play by agent 2. Agent 2 solves*

$$\max_{y(z)} E\left[-y - (z+\delta)^2\right]; \; z(\varepsilon) = x^*(z) + y(z) + \varepsilon$$

where $x^*(z)$ *is some fixed play by agent 1.*

Theorem 6 *There is no FNE in Problem 5.*

PROOF. Substitute the constraint into agent 1's problem for

$$\max_{z(\varepsilon)} E\left[y^*(z) - z - (z-\delta)^2 + \varepsilon\right].$$

The maximising variable is now written as $z(\varepsilon)$ as, for fixed $y^*(z)$, selecting $x(z)$ and $z(\varepsilon)$ are equivalent by equation 1. The problem may now be written as

$$\max_{z}\left[y^*(z) - z - (z-\delta)^2\right]; \tag{2}$$

by first moving the uncontrollable ε outside of the maximand and noting that the expectations operator may be dropped in the absence of stochastic terms.

As the problem is now deterministic it is solved by some constant $z(\varepsilon)$. Let this constant be ζ so that, by equation 1,

$$\zeta = x(\zeta) + y(\zeta) + \varepsilon;$$

a contradiction. As agent 1 has no optimal strategy, there can be no FNE. □

This result generalises to other problems requiring a constant maximum.

Treating x and y as functions of z is different than treating them as functions of ε, although, by Lemma 2, functions of z can be expressed as functions of ε in equilibrium. Reconsider Problem 5 by assuming x and y to be functions of ε. Equation 2 becomes

$$\max_{z}\left[-z - (z-\delta)^2\right];$$

which is solved by the constant $\zeta \equiv \delta - \frac{1}{2}$. There is, however, no contradiction now:

$$x(\varepsilon) = \zeta - y^*(\varepsilon) - \varepsilon.$$

4 Existence: payoffs quadratic in control

An existence result is now presented by slightly perturbing the model above. The commons game interpretation now becomes clearer. Agents set x and y, levels of industrial output, which produce greenhouse gas emissions in fixed proportions as a side-effect. These contribute to an atmospheric stock variable,

z. Agents' utility is quadratic both in output, possibly consistent with a labour - leisure tradeoff, and in atmospheric GHG content.

Problem 7 (quadratic) *Agents 1 and 2 have objective functions*

$$u = -x^2 - (z-\delta)^2\,; \tag{3}$$
$$v = -y^2 - (z+\delta)^2\,; \tag{4}$$

respectively, where $\delta \in \Re_+$ is a parameter. Agent 1 solves

$$\max_{x(z)} E\left[-x^2 - (z-\delta)^2\right];\ z(\varepsilon) = x(z) + y^*(z) + \varepsilon$$

where $y^*(z)$ is some fixed play by agent 2. Agent 2 solves

$$\max_{y(z)} E\left[-y^2 - (z+\delta)^2\right];\ z(\varepsilon) = x^*(z) + y(z) + \varepsilon$$

where $x^*(z)$ is some fixed play by agent 1.

Substituting the constraint into agent 1's problem produces

$$\max_{z(\varepsilon)} E\left[-2z^2 + 2y^*z + 2\varepsilon z - (y^*)^2 - 2\varepsilon y^* - \varepsilon^2 + 2\delta z - \delta^2\right]. \tag{5}$$

The multiplicative terms in ε prevent this problem being rewritten as in Theorem 6 (attempts to use the constraint to remove the ε terms merely re-introduce the $x(z)$ terms). Problem 7 may therefore have a solution. The multiplicative terms also prevent removal of the expectations operator as was done in equation 2, above. Hence, the problem is:

$$\max_{z(\varepsilon)} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left[-2z^2 + 2y^*z + 2\varepsilon z - (y^*)^2 - 2\varepsilon y^* - \varepsilon^2 + 2\delta z - \delta^2\right] f(\varepsilon)\, d\varepsilon. \tag{6}$$

Again, regard agent 1 as controlling $z(\varepsilon)$ when y^* is fixed. A necessary condition for a maximum is:

$$\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} [-4z + 2y^{*\prime}z + 2y^* + 2\varepsilon - 2y^*y^{*\prime} - 2\varepsilon y^{*\prime} + 2\delta]\, f(\varepsilon)\, d\varepsilon = 0. \tag{7}$$

The injective relationship demonstrated in Lemma 2 means that agent 1's ability to condition its play on z effectively allows conditioning on ε. Expected utility maximisation and pointwise maximisation by realisations of ε

are therefore equivalent. Necessary condition 7 must thus hold for all ε as there is otherwise an $\hat{\varepsilon} \in [-\bar{\varepsilon}, \bar{\varepsilon}]$ such that $z(\hat{\varepsilon})$ is not a stationary point. Therefore, as an extremum requires that the contents of the square bracketed term in equation 7 equal zero for all feasible ε:

$$y^{*\prime} = 1 + \frac{z - \delta}{x^*}. \tag{8}$$

Similarly, for agent 2,

$$x^{*\prime} = 1 + \frac{z + \delta}{y^*}. \tag{9}$$

Having implicitly assumed that $x, y \in \mathcal{C}^1$, equations 8 and 9 make them elements of $\mathcal{C}^\infty$ as well.

Differentiating the square bracketed term in equation 7 with respect to z a second time and noting that $z(\varepsilon)$ must maximise for any realisation of $\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$ produces a necessary condition for a maximum:

$$-2 + 2y^{*\prime} + x^* y^{*\prime\prime} - \left(y^{*\prime}\right)^2 \leq 0. \tag{10}$$

For agent 2, the equivalent requirement is that

$$-2 + 2x^{*\prime} + y^* x^{*\prime\prime} - \left(x^{*\prime}\right)^2 \leq 0. \tag{11}$$

Note that the first and second order conditions are independent of the support of ε. Indeed, they are obtained even when $\varepsilon \equiv 0$.

Differentiate equations 8 and 9 for

$$y^{*\prime\prime} = \frac{1}{x^*} - \frac{z - \delta}{\left(x^*\right)^2} x^{*\prime}; \text{ and } x^{*\prime\prime} = \frac{1}{y^*} - \frac{z + \delta}{\left(y^*\right)^2} y^{*\prime};$$

and substitute them into equations 10 and 11 for

$$\frac{z - \delta}{x^*} \left\{ 1 + \frac{z - \delta}{x^*} + \frac{z + \delta}{y^*} \right\} \geq 0; \text{ and} \tag{12}$$

$$\frac{z + \delta}{y^*} \left\{ 1 + \frac{z - \delta}{x^*} + \frac{z + \delta}{y^*} \right\} \geq 0. \tag{13}$$

A maximum therefore requires that $\frac{z-\delta}{x}$, $\frac{z+\delta}{y}$ and $\left\{1+\frac{z-\delta}{x}+\frac{z+\delta}{y}\right\}$ have the same sign $\forall z$. Thus, $sign\left[x^*\left(z\right)\right]=sign\left[y^*\left(z\right)\right]\Leftrightarrow z\notin\left(-\delta,\delta\right)$: agents 'pull' in opposite 'directions' when $z\in\left(-\delta,\delta\right)$ and in the same 'direction' otherwise.

As equations 8 and 9 may not have closed form solutions, the linear and symmetric special cases mentioned above are now examined.

4.1 Linear solutions

Lemma 8 *Equations 8 and 9 have only two linear solutions. These correspond to*

$$x^*=\left(z-\delta\right)a; \quad (14)$$
$$y^*=\left(z+\delta\right)a; \quad (15)$$

where a is a constant implicitly defined by

$$a^2-a-1=0. \quad (16)$$

Denote the roots of equation 16 by $a_+>0>a_-$. There is thus a form of symmetry between the agents' strategies: x^* is the function of $z-\delta$ that y^* is of $z+\delta$.

PROOF. Substitute the general formulation for linear strategies of z,

$$x=az+b;\ \text{and}\ y=cz+d;$$

where a,b,c and d are real constants, into equations 8 and 9 for

$$\left(ac-c-1\right)z=d\left(1-a\right)+\delta=b\left(1-c\right)-\delta.$$

As this must hold for all $z, c=a, a^2-a-1=0$ and $b=-d=\frac{\delta}{1-a}$, producing equations 14 and 15.

As both roots of a satisfy the second order conditions of equations 12 and 13, these are maxima.

□

As solutions to differential equations 8 and 9 introduce constants of integration, they are not necessarily solutions to the original maximisation Problem 7. The linear solutions are:

Theorem 9 *The only strategies to support a FNE in Problem 7 when strategies are restricted to be linear are the two pairs of symmetric strategies identified in Lemma 8.*

PROOF. Substituting the expression for y^* from Lemma 8 into the maximand in agent 1's problem (equation 5) produces

$$\max_{z(\varepsilon)} E\left[\left(2a-2-a^2\right)z^2+2\left(\left(a\delta+\varepsilon\right)\left(1-a\right)+\delta\right)z-a^2\delta^2-2a\delta\varepsilon-\varepsilon^2-\delta\right].$$

By Lemma 8, $x^*=(z-\delta)\,a$ is an extremum of the maximand. When a is defined according to equation 16 the maximand is concave in x given y^* and vice versa.

There are no other linear candidates as Lemma 8 identified all linear candidate solutions to Problem 7. $\square$

Therefore:

Theorem 10 *The expected payoffs to the linear equilibria in quadratic Problem 7 are*

$$E\left[u\right]=E\left[v\right]=-\left(1+a^2\right)\left(\delta^2+\frac{1}{5}E\left[\varepsilon^2\right]\right).$$

PROOF. Substitution of the linear strategies into the objective functions produces

$$\begin{aligned}E\left[u\right]&=-\left(1+a^2\right)\left(\delta^2-2\delta E\left[z\right]+E\left[z^2\right]\right);\ \text{and}\\ E\left[v\right]&=-\left(1+a^2\right)\left(\delta^2+2\delta E\left[z\right]+E\left[z^2\right]\right).\end{aligned}$$

Equation 1, which defined the state variable, and the linear strategies allow calculation of $z=\frac{\varepsilon}{1-2a}$. Therefore $E\left[z\right]=0$ and, by equation 16, $E\left[z^2\right]=\frac{1}{5}E\left[\varepsilon^2\right]$. The result follows. $\square$

Although the downward sloping linear path yields a higher payoff than does the upward sloping one, the linear solutions are similar in the following respect.

The objective functions are such that each agent would like to set its control to zero and would like $z = 0$. Their strategies are therefore chosen to offset the consequences of a realisation of $\varepsilon \neq 0$. Both strategies produce the same variance for z, a variance less than that of ε.

Other strategies supporting equilibria may be constructed by using the linear symmetric equilibrium strategies and varying them beyond the supportable domain. These variations do not alter expected utility.

4.2 Symmetric agents and play

Two special cases of symmetric play are now considered. In the first, Problem 7 is restricted by setting $\delta = 0$ and requiring that $x(z) = y(z)$. This second restriction is removed in the next section. The present restrictions reduce 8 and 9 to

$$x^{*\prime} = 1 + \frac{z}{x^*}. \tag{17}$$

Lemma 8 provides the linear solutions to equation 17 as special cases; these are plotted in Figure 1. Representative non-linear solutions are drawn by noting that $x = -z$ defines the horizontal isocline ($x^{*\prime} = 0$) while $x^* = 0$ defines the vertical ($x^{*\prime} = \pm\infty$).

As the linear solutions are addressed by Theorem 9, they support FNE. Consider therefore the non-linear solutions to equation 17. Second order condition 10 reduces here to

$$-2\left[(1 - x^{*\prime})^2 - x^* x^{*\prime\prime} + 1\right] \leq 0; \tag{18}$$

or, by reducing equation equation 12 for:

$$(2z + x^*)\, z \geq 0. \tag{19}$$

The $z = 0$ and $x = -2z$ lines therefore divide (z, x) space into four regions, two satisfying the second order conditions and two violating them.

Lemma 11 *The symmetric non-linear solutions to equation 17 that cross $x =$ 0 are not admissible strategy functions.*

PROOF. Figure 1 reveals that these $x(z) = y(z)$ are neither defined over all $z \in \Re^*$ nor are they functions. Such strategy pairs therefore violate the definition of admissible function pairs. □

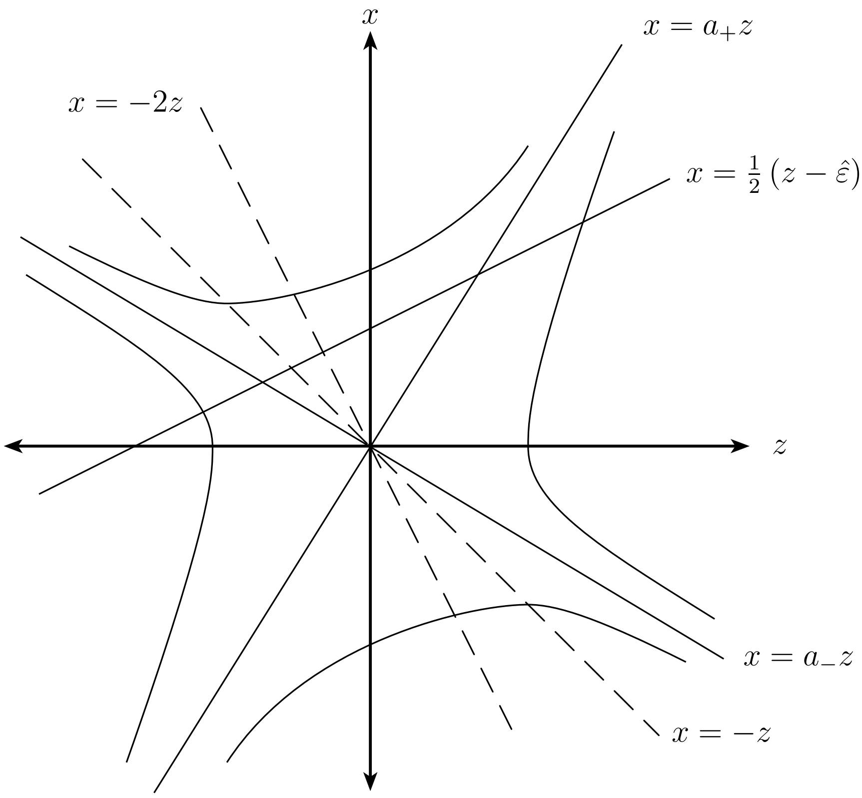

Fig. 1. Solutions to equation 17

Lemma 12 *The symmetric non-linear solutions to equation 17 that cross $z = 0$ are not admissible strategy functions.*

PROOF. Solutions crossing the $z = 0$ locus violate condition 19 in doing so. $\square$

As the previous two lemmata eliminate all non-linear candidates:

Theorem 13 *No symmetric strategy pairs that are non-linear over the supportable domain support a FNE in Problem 7 when $\delta = 0$.*

Lemma 11 holds independently of the distribution of ε, including the degenerate: agents must be able to form conjectures of others' play for all $z \in \Re^*$.

The candidates eliminated by Lemma 12 are also eliminated independently of ε's distribution. A realisation of the stochastic variable, $\hat{\varepsilon}$, defines, through state equation 1, the line $x = \frac{1}{2}(z - \hat{\varepsilon})$, as illustrated in Figure 1. Admissibility requires candidates to intersect the line once for any $\hat{\varepsilon}$ in the support of ε. None of the non-linear candidates crossing $z = 0$ do for any distribution of ε.

The preceding results may be generalised slightly. If $\delta \neq 0$ but both agents had the identical Euler condition

$$x^{*\prime} = 1 + \frac{z+\delta}{x^*};$$

then the diagram in Figure 1 is simply translated.

4.3 Symmetric agents and asymmetric play

Continue to restrict Problem 7 by setting $\delta = 0$ but now allow $x(z) \neq y(z)$, so that agents may play asymmetrically.[5] The change of variables

$$\xi(z) \equiv \frac{z}{x}; \text{ and } \eta(z) \equiv \frac{z}{y}; \tag{20}$$

produce

$$x' = \frac{1-\xi' x}{\xi}; \text{ and } y' = \frac{1-\eta' y}{\eta};$$

so that the first order conditions in equations 8 and 9 may be written as

$$\eta' = \frac{\eta(1-\eta-\xi\eta)}{z}; \text{ and } \xi' = \frac{\xi(1-\xi-\xi\eta)}{z}; \tag{21}$$

so that

$$\frac{d\eta}{d\xi} = \frac{\eta(1-\eta-\xi\eta)}{\xi(1-\xi-\xi\eta)}. \tag{22}$$

One solution to this is $\xi = \eta$, which becomes $x(z) = y(z)$ when the change of variables is reversed. This has already been considered.

Figure 2 contains the phase diagram associated with equation 22. The flowlines indicate five stationary points at which $\xi' = \eta' = 0$:

$$\mathcal{S} \equiv \{(\xi,\eta) \,|\, (\xi,\eta) \in \{(0,0),(0,1),(1,0),(-a_-,-a_-),(-a_+,-a_+)\}\}.$$

[5] Klemperer and Meyer (1989, Proposition 3) demonstrates the non-existence of asymmetric equilibria. As that proof relies on the behaviour of their differential equation system and on their assumptions about behaviour when firm profits are negative, the result does not apply directly to the present situation.

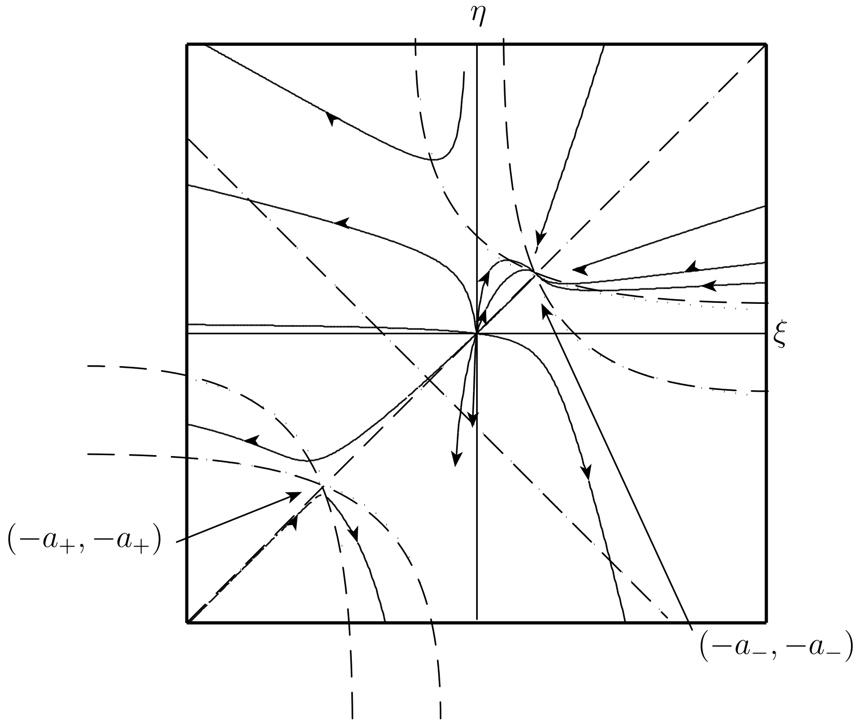

Fig. 2. Phase diagram in (ξ, η) space

The curved dashed lines represent the horizontal and vertical isoclines, those points at which $\frac{d\eta}{d\xi} = 0$ or $\frac{d\eta}{d\xi} = \pm\infty$, respectively.

The downward sloping diagonal dashed line is $\xi + \eta = -1$, one of the boundaries for the second order conditions of equations 12 and 13. These are satisfied in the first quadrant and in the third below the diagonal line.

The linear solutions found by Lemma 8 are now the points $(\xi, \eta) = (-a, -a)$, where $a = \{a_-, a_+\}$, which lie at the two intersections of the horizontal and vertical isoclines. By equation 22, asymmetric paths through them must satisfy either $\frac{d\eta}{d\xi} = \pm 1$. To understand why, note that

$$\lim_{\xi \to \eta} \frac{\eta'}{\xi'} = \frac{\xi'}{\eta'};$$

so that $(\xi')^2 = (\eta')^2$ as well.

The interpretation of a path in (ξ, η) space is slightly more subtle than it is usually. In (x, y, z) space, any z coordinate can be assigned to any (x, y) pair. Thus, transformation into (ξ, η) space means that any (ξ, η) is consistent with any non-zero and finite z (the case of $z \in \{0, \pm\infty\}$ is addressed below). Thus, each path in (ξ, η) space represents an infinite number of paths in (x, y, z) space, the paths being indexed by non-zero and finite z. The same (ξ, η) point may therefore represent both $(\xi(z), \eta(z))$ and $(\xi(-z), \eta(-z))$ for any non-zero and finite z. Therefore, in Figure 2, the flowlines indicate decreasing values of z when $z < 0$. [6]

Only a few points in (ξ, η) space are consistent with $z \in \{0, \pm\infty\}$. As admissible $x(z)$ and $y(z)$ must be defined for all $z \in \Re^*$, candidate FNE strategies must contain these points.

Lemma 14 *Candidate FNE solutions must contain an element of $\mathcal{S}$.*

The proof establishes that the only (ξ, η) points consistent with $z = 0$ are those in $\mathcal{S}$.

PROOF. When $z = 0$ but $x, y \neq 0$, the change of variables defined in equations 20 implies $(\xi, \eta) = \vec{0}$, an element of $\mathcal{S}$. When $z = x = 0$ but $y \neq 0$, L'Hôpital's rule and equation 9 imply that $\xi = \frac{1}{x'} = \frac{y}{y+z} = 1$ while $\eta = 0$, as above. This produces $(\xi, \eta) = (1, 0)$, another element of $\mathcal{S}$. Symmetry of ξ and η produces the third element of $\mathcal{S}$ when $y = z = 0$ but $x \neq 0$.

When $x = y = z = 0$, L'Hôpital's rule again provides $\xi = \frac{1}{x'}$ and $\eta = \frac{1}{y'}$. But as $x = y$ it must be that $\xi = \eta$ so that $x' = y'$ as well. Therefore, first order conditions 8 and 9 become $x' = 1 + \frac{1}{x'}$ so that $x' = a$. Hence $(\xi, \eta) = \left(\frac{1}{a}, \frac{1}{a}\right)$ which, by definition of a, produces the final two elements of $\mathcal{S}$, $(-a, -a)$. $\square$

Similarly, by considering those points consistent with infinite z:

Lemma 15 *Candidate FNE solutions must satisfy one of the following conditions:*

(1) either one of ξ or η becomes infinite as z does; or
(2) $(\xi, \eta) \to (-a, -a)$ as z becomes infinite.

PROOF. Admissible strategies must be defined for $z = \pm\infty$. By equations 20, infinite z implies infinite ξ and η for any finite x and y.

When x is finite but y infinite, $\xi \to \pm\infty$ but $\eta = \frac{z}{y}$ so that L'Hôpital's rule is invoked for

$$\eta = \frac{1}{y'} = \left(1 + \frac{z}{x}\right)^{-1} = \frac{1}{1+\xi} = 0.$$

A similar result holds when x is infinite but y finite.

[6] Replacing z with $-z$ such that $\xi(z) = \xi(-z)$ and $\eta(z) = \eta(-z)$ changes the signs in equations 21 but not in equation 22.

The preceding motivates the first condition of the lemma; the second is now addressed. When x and y are both infinite, L'Hôpital's rule now produces

$$\eta = \frac{1}{1+\xi} \text{ and } \xi = \frac{1}{1+\eta};$$

so that $\eta + \xi\eta = \xi + \xi\eta = 1$. Therefore $\xi = \eta$ and $1 - \xi - \xi^2 = 0$ so that $(\xi, \eta) = (-a, -a)$. □

These lemmata shed some light on the two linear solutions to Problem 7 found by Lemma 8: the points $(\xi, \eta) = (-a, -a)$ are the only points in (ξ, η) space that are consistent with all $z \in \Re^*$. Furthermore, if z is finite and non-zero at either of these points, $\xi' = \eta' = 0$ and the 'path' necessarily remains at the point.

As candidate strategies must also avoid regions in (ξ, η) space that violate the second order conditions (the second and fourth quadrants and the third above $1 - \xi - \eta = 0$), FNE strategies may only be taken from the following families:

$\mathcal{D}_1$ the two linear FNE;
$\mathcal{D}_2$ the continuum of paths converging on $(-a_-, -a_-)$ from above;
$\mathcal{D}_3$ the continuum of paths converging on $(-a_-, -a_-)$ from the origin;
$\mathcal{D}_4$ the two paths converging on $(-a_-, -a_-)$ from $(0, 1)$ and $(1, 0)$, respectively; and
$\mathcal{D}_5$ the two paths diverging asymmetrically from $(-a_+, -a_+)$.

The following lemmata eliminate all but the $\mathcal{D}_1$ family from consideration.

Lemma 16 *None of the $\mathcal{D}_2$ paths support FNE.*

PROOF. These paths cannot be parameterised by $z \in \Re^*$. Suppose, for example, that $z = -\infty$ at $(\xi, \eta) = (\infty, \infty)$. The flowlines prevent (ξ, η) returning to $(-a_-, -a_-)$ as z increases to zero. The reverse also holds: if $z = \infty$ at $(\xi, \eta) = (\infty, \infty)$ the flowlines again prevent (ξ, η) returning to $(-a_-, -a_-)$ as z increases to zero. □

The next two lemmata use a similar argument. They find that the paths considered either behave as above or become discontinuous (and are therefore not solutions to the original differential equations).

Lemma 17 *None of the $\mathcal{D}_3$ paths support FNE.*

PROOF. As $(\xi,\eta) = \vec{0}$ is consistent with $z = 0$, but not with $z = \pm\infty$, the path considered contains the points $(\xi(-\infty), \eta(-\infty)) = (-a_-, -a_-)$, $(\xi(0), \eta(0)) = \vec{0}$ and $(\xi(\infty), \eta(\infty)) = (-a_-, -a_-)$. By Lemma 14, both $x(0)$ and $y(0)$ are non-zero. In (ξ, η) space, $(\xi(\hat{z}), \eta(\hat{z})) = (\xi(-\hat{z}), \eta(-\hat{z}))$ for any given $\hat{z}$. Therefore, $\frac{z}{x(z)} = \frac{-z}{x(-z)}$ so that $x(z) = -x(-z)$. As $x(0) \neq 0$ (and $y(0) \neq 0$) this is discontinuous at $z = 0$ and therefore inconsistent with equations 8 and 9. □

Lemma 12 and Figure 1, which addressed the case of symmetric play, illustrate the argument: $x(z) = -x(-z)$ implies a symmetric solution jumping from one path to its mirror image in the horizontal axis at $z = 0$. Those paths continuous at $x(0)$ may also be seen in Figure 1.

Lemma 18 *Neither of the $\mathcal{D}_4$ paths support FNE.*

PROOF. The proof here differs from that of Lemma 17 in one respect: $x(0)$ is still distinct from zero, but $y(0) = 0$. L'Hôpital's rule, though, shows that $x'(0) = 1 + \frac{1}{y'(0)}$. As $y'(0) = 1 + \frac{z}{x(0)} = \frac{x(0)+z}{x(0)} = 1$, $x'(0) = 2$. As $x'(0)$ is well defined, the solution must still be continuous. As $x(0) \neq 0$, the $\mathcal{D}_4$ paths are not. □

Lemma 19 *Neither of the $\mathcal{D}_5$ paths support FNE.*

PROOF. Consider $\mathcal{D}_5$ paths at $\xi = \eta$. Here $x = y = z = 0$ allowing the ODEs to be written

$$x'(0)\, y'(0) = 1 + x'(0) = 1 + y'(0)$$

so that $x'(0) = y'(0)$, a symmetric solution. As the $\mathcal{D}_5$ paths are not symmetric, they do not satisfy the original differential equations. □

Having eliminated all candidates:

Theorem 20 *When $\delta = 0$, Problem 7 has no FNE that is non-linear over the supportable domain.*

Thus, while the support of ε plays a role in refining the equilibrium set, it never affects play. In Klemperer and Meyer (1989) it plays a more substantive role.

In the supportable domain, the only linear solutions are symmetric. These are more efficient than asymmetric linear functions would be as they allow agents to split the task, and quadratic cost, of offsetting the ε shocks which cause z to deviate from zero.

5 The Pareto frontier

Now trace the Pareto frontier is traced for Problem 7. Weight objective functions 3 and 4 by $\phi, (1-\phi) \in [0,1]$, respectively, so that social welfare is

$$\begin{aligned} w &= \phi u + (1-\phi)\, v \\ &= -\phi x^2 - \phi\,(z-\delta)^2 - (1-\phi)\, y^2 - (1-\phi)\,(z+\delta)^2 . \end{aligned}$$

As the planner maximises expected social welfare subject to equation 1, it must

$$\begin{aligned} &\max_{x,y} E\,[w] \\ &= \max_{x,y} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \Big[-\phi \left(2x^2 + y^2 + \delta^2 + \varepsilon^2 + 2xy + 2\varepsilon\,(x+y) - 2\delta\,(x+y+\varepsilon) \right) \\ &\quad - (1-\phi) \left(x^2 + \delta^2 + \varepsilon^2 + 2\,(y+\varepsilon)\,(x+y) + 2\delta\,(x+y+\varepsilon) \right) \Big]\, f(\varepsilon)\, d\varepsilon. \end{aligned}$$

Differentiating with respect to z, as before, is possible as the social planner controls $z(\varepsilon)$. This only produces a single first order condition. Continue, therefore, to regard the choice variables as x and y. As the gradients in these must be zero for all realisations of ε, necessary conditions are derived by differentiating the square bracketed expression to obtain the Euler conditions.

The first order conditions are therefore

$$x^* = \frac{1}{\phi}\,[(2\phi-1)\,\delta - z] \text{ and } y^* = \frac{1}{1-\phi}\,[(2\phi-1)\,\delta - z] .$$

As the maximand is concave in x and y, these indicate a maximum.

Equation 1 and substitution using the assumption that $E\,[\varepsilon] = 0$ produces the individual payoffs:

$$E\,[u] = -\frac{(1-\phi)^2\,(1+\phi^2)}{[\phi\,(1-\phi)+1]^2} \left[5\delta^2 + E\left[\varepsilon^2\right] \right]; \text{ and} \tag{23}$$

$$E\,[v] = -\frac{\phi^2 \left(1 + (1-\phi)^2 \right)}{[\phi\,(1-\phi)+1]^2} \left[5\delta^2 + E\left[\varepsilon^2\right] \right]. \tag{24}$$

Together equations 23 and 24 define a curve parameterised by ϕ. In $(E[u], E[v])$ space, the curve is concave and symmetric about $E[u] = E[v]$. Furthermore, it is tangent to the horizontal axis ($E[v] = 0$) at $\phi = 0$ and to the vertical axis ($E[u] = 0$) at $\phi = 1$.

In the case of symmetric agents ($\delta = 0, \phi = \frac{1}{2}$), expected payoffs simplify to

$$E[u] = E[v] = -\frac{1}{5}E\left[\varepsilon^2\right]. \tag{25}$$

5.1 Comparison with the game

As expected, the payoffs to the planned outcome in equation 25 are greater than those to the game (presented in Theorem 10).

The case of symmetric agents ($\delta = 0, \phi = \frac{1}{2}$) also allows comparison of the responsiveness of the optimal controls, $\frac{\partial x}{\partial \varepsilon}$. In the game, $x = za$ and $z = \frac{\varepsilon}{1-2a}$ so that $\frac{\partial x}{\partial \varepsilon} = \frac{a}{1-2a} = \left\{-\frac{1+\sqrt{5}}{2\sqrt{5}}, \frac{1-\sqrt{5}}{2\sqrt{5}}\right\}$. In the planned environment, $x = -2z$ and $z = \frac{1}{5}\varepsilon$. Therefore $\frac{\partial x}{\partial \varepsilon} = -\frac{2}{5}$, intermediate to those of the game.

In all cases, $\frac{\partial x}{\partial \varepsilon} < 0$, which has a sensible interpretation: in the symmetric case, both agents wish to set their controls to zero and would like the state to be zero. Without shocks, this could be achieved; therefore shocks are to be counteracted. That the response of the planned outcome is intermediate suggests that the game responses reflect the failure of the game to attain the planned outcome.

6 FNE with transfers

Now augment agents' (symmetric) objective functions to define a new problem:

Problem 21 (transfer) *Agents 1 and 2 have objective functions*

$$\begin{aligned} u &= -x(z)^2 - z^2 - r(y) + s(x); \\ v &= -y(z)^2 - z^2 + r(y) - s(x); \end{aligned}$$

where $x(z)$ and $y(z)$ are as above but $r(y) \geq 0$ is now a transfer controlled by agent 1 and $s(x) \geq 0$ one controlled by agent 2.

This formulation therefore allows utility transfers. Clearly this does not correspond as closely to the motivating problem as would a model of transfers in consumption goods. The formulation is adopted, though, for analytical tractability. Nevertheless, refer to x and y as emission functions and to r and s as transfer functions.

Extending the previous approach, so that agent 1 chooses an $x(z)$ and a $r()$ to maximise $E[u]$ against 2's fixed $y(z)$ and $s()$, no longer suffices: a fixed $y^*(z)$ only responds to changes in z. If z is not influenced by transfers, then neither are emissions. Two approaches therefore present themselves as possible. The first involves specifying $x(r-s,z)$ and $y(r-s,z)$. Making emissions a function of the transfers is appealing: it regards the quantity possessed of the transferable resource as a second payoff relevant state variable. This approach is not pursued to avoid the complications of PDEs.

The second involves considering a three stage game in fictional time in which agents first announce and commit to their transfer functions. They next calculate and commit to optimal $x()$ and $y()$; finally, ε is realised. This is, of course, an *ad hoc* formulation, converting the game into a form of Stackelberg problem. This formulation also assumes a commitment technology. Nevertheless, this approach is pursued as an initial attempt at the problem.

First, though, what transfers are required to support the first best?

6.1 Implementing the symmetric first best

The calculations in Section 5 show that the first best is supported by emissions of the form $x^* = y^* = -2z$ when agents are symmetric. As these do not support an FNE of the game without transfers, it is now asked what transfers are capable of supporting these emission functions. The transfer functions are not here required to support an equilibrium.

In the second stage, agents regard the transfers $r(y)^*, s(x)^*$ and the other's x or y as fixed. Agent 1 therefore must

$$\max_z E\left[-x(z)^2 - z^2 - r^*(y^*(z)) + s^*(x(z))\right] \text{ s.t. } \begin{aligned} z &= x(z) + y^*(z) + \varepsilon \\ y^* &= -2z; \end{aligned}$$

which has first order conditions [7]

$$-6(3z-\varepsilon) - 2z + 2r^{*\prime}(-2z) + 3s^{*\prime}(3z-\varepsilon) = 0.$$

[7] As $r(y), s(x) \geq 0$ these may be non-differentiable at certain points. The solutions that are eventually derived are differentiable for all z.

Thus, as $x^* = y^* = -2z$ and $r^* = s^*$, this reduces to

$$5r^{*\prime}(y) = 5y;$$

so that

$$r^*(y) = \frac{1}{2}y^2 + c. \tag{26}$$

The non-negativity requirement is satisfied for any $c \geq 0$. As $c > 0$ simply imposes an additional cost on the transferring agent without altering the recipient's incentives, assume that

$$c = 0. \tag{27}$$

This is a counter-intuitive form for a transfer function, paying out as the other agent's emissions deviate from the ideal of $y = 0$.

Expected utility may now be calculated. The functional forms set $z = \frac{1}{5}\varepsilon$ so that $x^* = y^* = -\frac{2}{5}\varepsilon$ and $r^* = s^* = -\frac{2}{25}\varepsilon^2$. Therefore

$$E[u] = E[v] = -\frac{1}{5}E\left[\varepsilon^2\right];$$

the same as the planned outcome. The transfer function in equation 26 therefore maximises the objective functions.

6.2 FNE in the three stage game

It is naturally of interest to ask whether the above emissions and transfer scheme support an equilibrium. Noting that it has linear emissions and quadratic transfer functions, functions are restricted to these classes in the following section. More general transfer functions are considered after that. These sections find that, when functions are so restricted, there is an FNE with linear emissions and quadratic transfers (although not the first best above). The final section shows that, when emissions are constrained to be linear, the optimal transfers are quadratic but conjectures that the linear-quadratic result does not generally hold when both functions are not constrained.

6.2.1 Linear emissions and quadratic transfers

Equilibria when emissions functions are constrained to be linear, $x(z) = \xi z$ and $y(z) = \eta z$, and transfer functions quadratic, $r(y) = \rho y^2$ and $s(x) =$

$\sigma x^2, \rho, \sigma \geq 0$, are now sought.

6.2.1.1 The second stage In the second stage, agents must

$$\max_x \left\{ E\left[-x^2 - z^2 - \rho^* \left(y^*\right)^2 + \sigma^* x^2 \right] \text{ s.t. } z = x + y^* + \varepsilon \right\}; \text{ and}$$
$$\max_y \left\{ E\left[-y^2 - z^2 + \rho^* y^2 - \sigma^* \left(x^*\right)^2 \right] \text{ s.t. } z = x^* + y + \varepsilon \right\}.$$

Proceeding as before yields the first order conditions

$$\rho\eta^2 + (\sigma - 1)\,\xi\eta - (\sigma - 1)\,\xi + 1 = 0; \text{ and} \tag{28}$$
$$\sigma\xi^2 + (\rho - 1)\,\xi\eta - (\rho - 1)\,\eta + 1 = 0; \tag{29}$$

when $z \neq 0$. Equations 28 and 29 produce lengthy expressions for $\xi\,(\rho, \sigma)$ and $\eta\,(\rho, \sigma)$ that are not amenable to interpretation. Note that $(\xi = \eta = -2) \Rightarrow \left(\rho = \sigma = \frac{1}{2}\right)$, the first best of Section 6.1.

The second order conditions require that

$$(\sigma - 1)\,(1 - \eta)^2 - 1 - \rho\eta^2 \leq 0; \text{ and} \tag{30}$$
$$(\rho - 1)\,(1 - \xi)^2 - 1 - \sigma\xi^2 \leq 0; \tag{31}$$

respectively.

6.2.1.2 The first stage Consider now the agents' problem in the first stage. Let $\xi\,(\rho, \sigma)$ and $\eta\,(\rho, \sigma)$ be candidate best response correspondences, therefore solving equations 28 and 29. Given these, it is necessary that $\frac{\partial E[u]}{\partial \rho}|_{\sigma^*} = \frac{\partial E[v]}{\partial \sigma}|_{\rho^*} = 0$. Working with agent 1,

$$E\,[u] = E\left[-\xi^2\,(\rho, \sigma)\, z^2 - z^2 - \rho\eta^2\,(\rho, \sigma)\, z^2 + \sigma\xi^2\,(\rho, \sigma)\, z^2 \right], \tag{32}$$

where

$$z = \frac{\varepsilon}{1 - \xi\,(\rho, \sigma) - \eta\,(\rho, \sigma)}.$$

Differentiating with respect to ρ and assuming that $1 - \xi - \eta \neq 0$ then produces a necessary condition for the first stage,

$$\begin{aligned} 0 = {} & (1 - \xi - \eta)\left[-2\xi\xi_\rho - \eta^2 - 2\rho\eta\eta_\rho + 2\sigma\xi\xi_\rho\right] \\ & +2\,(\xi_\rho + \eta_\rho)\left[(\sigma - 1)\,\xi^2 - 1 - \rho\eta^2\right]. \end{aligned} \tag{33}$$

The equivalent expression for agent 2 is

$$
\begin{aligned}
0 = &(1-\xi-\eta)\left[-2\eta\eta_\sigma - \xi^2 + 2\rho\eta\eta_\sigma - 2\sigma\xi\xi_\sigma\right] \\
&+2\left(\xi_\sigma+\eta_\sigma\right)\left[(\rho-1)\eta^2 - 1 - \sigma\xi^2\right]. \qquad (34)
\end{aligned}
$$

Calculations in Appendix A yield agent 1's second order conditions:

$$
\begin{aligned}
\frac{\partial^2 E[u]}{\partial\rho^2} = &\left[3\left(\xi_\rho+\eta_\rho\right)^2 + \left(\xi_{\rho\rho}+\eta_{\rho\rho}\right)[1-\xi-\eta]\right]\left[(\sigma-1)\xi^2 - 1 - \rho\eta^2\right] \qquad (35)\\
&+\left\{\left(\xi_{\rho\rho}+\eta_{\rho\rho}\right)[1-\xi-\eta] + 2\left(\xi_\rho+\eta_\rho\right)^2\right\}\left[4(\sigma-1)\xi\xi_\rho - 4\rho\eta\eta_\rho - \eta^2\right] \qquad (36)\\
&+[1-\xi-\eta]^2\left[(\sigma-1)\left(\xi_\rho^2+\xi\xi_{\rho\rho}\right) - 2\eta\eta_\rho - \rho\left(\eta_\rho^2+\eta\eta_{\rho\rho}\right)\right] \leq 0; \qquad (37)
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_{\rho\rho} = &\frac{\xi_\rho^2}{1-\xi} + \frac{(1-\xi)\left[\eta^2 + 2(\rho+1)\eta\eta_\rho + (\sigma-1)\left(\xi_\rho\eta+\xi\eta_\rho\right)\right]}{\Delta(\rho,\sigma)} \\
&-\frac{\xi_\rho}{\Delta(\rho,\sigma)}\Delta_\rho; \text{ and} \\
\eta_{\rho\rho} = &\frac{\eta_\rho\Delta_\rho}{\Delta} - \frac{\eta_\rho^2}{\eta} \\
&-\frac{\eta\left[(\sigma+1)\xi_\rho\eta + \xi\eta_\rho + \eta^2 + 2(\rho-1)\eta\eta_\rho + (\sigma-1)\left(\xi_\rho+\eta_\rho\right)\right]}{\Delta}.
\end{aligned}
$$

when $1-\xi-\eta \neq 0$.

Symmetric conditions exist for agent 2's problem.

Lemma 22 *The symmetric first best identified in Section 6.1 does not form an FNE of Problem 21.*

PROOF. Substitution of $\rho = \sigma = \frac{1}{2}$ and $\xi = \eta = -2$ into equations 28, 29, 33 and 34 solves the first two but not the second two. □

6.2.1.3 Symmetric solutions Consider situations in which $\sigma = \rho$ and $\eta = \xi$. In these cases the equations 28, 29, 33 and 34 can be reduced to two equations. The first order conditions of the second stage reduce to

$$
(2\rho-1)\xi^2 - (\rho-1)\xi + 1 = 0; \qquad (38)
$$

while those of the first stage become

$$2\left[(1-2\rho)\,\xi^2+\rho\xi+1\right]\eta_\rho+(1-2\xi)\,\xi^2=0. \tag{39}$$

when equation 38 is satisfied.

Finally, the expression for η_ρ is derived from matrix A.1:

$$\eta_\rho=-\frac{\xi\left[2\rho\xi^2+(2\xi-1)\,(\rho-1)\right]}{4\rho\xi^2\,(2\rho-1)+(\rho-1)^2\,(2\xi-1)}. \tag{40}$$

A solution to equations 38 and 39 is obtained by manipulation. First rearrange equation 38 for

$$\rho\xi\,(2\xi-1)=\xi^2-\xi-1; \tag{41}$$

which, in turn, may be rearranged for

$$\begin{aligned}(\rho-1)\,\xi\,(2\xi-1)+\xi^2+1&=0;\ \text{or}\\ (2\rho-1)\,\xi\,(2\xi-1)+\xi^2+1-\rho\xi\,(2\xi-1)&=0.\end{aligned} \tag{42}$$

Equation 41 then simplifies this to

$$(2\rho-1)\,\xi\,(2\xi-1)+\xi+2=0. \tag{43}$$

Finally, the square bracketed term in equation 39 may be rearranged using equation 41 for

$$2\,[2+\xi]\,\eta_\rho=(2\xi-1)\,\xi^2. \tag{44}$$

Equation 40 may be divided by $2\xi-1$ as this cannot be a solution (substitution of it into equation 38 produces $\frac{5}{4}=0$); this produces

$$\begin{aligned}\frac{\eta_\rho}{2\xi-1}&=-\frac{\left[2\rho\xi^2+(2\xi-1)\,(\rho-1)\right]\xi}{4\rho\xi\,(2\xi-1)\,(2\rho-1)\,\xi+\left[(\rho-1)\,(2\xi-1)\right]^2}\\ &=\frac{2\rho\xi^3+(2\xi-1)\,\xi\,(2\rho-1)-\rho\xi\,(2\xi-1)}{4\rho\xi\,(\xi+2)-\left[(\xi^2+1)\,\xi^{-1}\right]^2};\end{aligned}$$

where the numerator has been expanded and the denominator simplified by equations 43 and 42, respectively. Now simplify the numerator by equations 41 and 43 and use equation 44 to replace the left hand side so that

$$\xi^2-2\xi-3=0.$$

Table 1
Expected utility for symmetric agents

Pareto rank	description	$E[u] = E[v]$	$E[z^2]$
1	first best	$-\frac{1}{5}E[\varepsilon^2]$	$\frac{1}{25}E[\varepsilon^2]$
2	transfer game, $x'(z) < 0$	$-\frac{2}{9}E[\varepsilon^2]$	$\frac{1}{9}E[\varepsilon^2]$
3	no transfer game, $x'(z) < 0$	$-\frac{5-\sqrt{5}}{10}E[\varepsilon^2]$	$\frac{1}{5}E[\varepsilon^2]$
4	transfer game, $x'(z) > 0$	$-\frac{2}{5}E[\varepsilon^2]$	$\frac{1}{25}E[\varepsilon^2]$
5	no transfer game, $x'(z) > 0$	$-\frac{5+\sqrt{5}}{10}E[\varepsilon^2]$	$\frac{1}{5}E[\varepsilon^2]$

Thus

$$(\rho, \xi) = \left\{ \left(\frac{1}{3}, -1\right), \left(\frac{1}{3}, 3\right) \right\}; \tag{45}$$

satisfy the first order conditions.

Both roots satisfy the second order conditions of the second stage, as identified by equations 30 and 31. As for those of the first stage, some tedious algebra determines that $(\rho, \sigma, \xi, \eta) = \left(\frac{1}{3}, \frac{1}{3}, -1, -1\right)$ sets equation 37 to $-\frac{1269}{16}$ while $(\rho, \sigma, \xi, \eta) = \left(\frac{1}{3}, \frac{1}{3}, 3, 3\right)$ sets it to $-\frac{218565}{16}$. Thus both real solutions are maxima. As $\rho = \sigma > 0$ in both cases, transfers are always non-negative. Hence:

Theorem 23 *The elements of equation 45 form a FNE of Problem 21 when emissions are constrained to be linear, and transfers quadratic.*

Equation 32 allows calculation of expected payoffs. For $(\rho, \sigma, \xi, \eta) = \left(\frac{1}{3}, \frac{1}{3}, -1, -1\right)$ these are $E[u] = E[v] = -\frac{2}{9}E[\varepsilon^2]$ while for $(\rho, \sigma, \xi, \eta) = \left(\frac{1}{3}, \frac{1}{3}, 3, 3\right)$ they are $E[u] = E[v] = -\frac{2}{5}E[\varepsilon^2]$. Thus the two FNE are Pareto ranked. Table 1 compares these expected payoffs to those arising from the game without transfers and from the social planner's solution.

As expected, the social planner's outcome Pareto dominates. Otherwise, while each equilibrium in the game with transfers is preferred to its counterpart in the game without transfers, agents' preferences over the games will depend on the probabilities assigned to each of the two FNE by an equilibrium selection mechanism.

One of the peculiar results in Table 1 is that expected variance of z for the FNE that sets x as a positive function of z is as low as that of the first best, and much less than its Pareto preferred alternative. This also stands in contrast to the expected variance of z in the game without transfers; in that case, both FNE yield the same result.

6.2.1.4 Asymmetric solutions Return to the general case in which symmetry may not hold by using NAG's hybrid Powell method root finder for nonlinear systems, `c05tbc`. This is used to simultaneously solve equations 28, 29, 33 and 34. Initial $(\rho, \sigma, \xi, \eta)$ seeds are randomly drawn from a uniform distribution over $(-10, 10)$. Most seeds fail to make progress between iterations, or exceed 200 iterations without finding roots. Repeating the programme until roots have been found successfully from 400 different seeds only identifies the two symmetric real roots found in equation 45. This suggests that there may not be asymmetric real equilibria to Problem 21.

Against this optimistic belief, Bézout's Theorem suggests that, in general, the number of complex intersections is the product of the degrees of the algebraic curves. In this case, as equations 28 and 29 are third order in ρ, σ, ξ and η and equations 33 and 34 seem to be of eleventh order, the product is 1089.

6.2.2 Linear emissions but general transfers

In this section, emission functions continue to be restricted to linear functions, but general transfers, $r(y), s(x) \geq 0$, are now allowed.

6.2.2.1 The second stage Without first imposing the non-negativity constraints, the problems at the second stage are to

$$\max_z E\left[-(z-y-\varepsilon)^2 - z^2 - r^*(y) + s^*(z-y-\varepsilon)\right]; \text{ and} \tag{46}$$

$$\max_z E\left[-(z-x-\varepsilon)^2 - z^2 + r^*(z-x-\varepsilon) - s^*(x)\right]. \tag{47}$$

These have first order conditions

$$\begin{aligned} &-2x(1-y') - 2z - r'y' + s'(1-y') = 0; \text{ and} \\ &-2y(1-x') - 2z + r'(1-x') - s'x' = 0. \end{aligned}$$

If $x(z)$ and $y(z)$ are linear functions, then it is necessary that

$$\begin{bmatrix} r'(y) \\ s'(x) \end{bmatrix} = \frac{2z}{(1-\xi-\eta)} \begin{bmatrix} \xi^2 - \xi^2\eta - \eta^2 + \xi\eta^2 - \xi\eta + \xi + 1 \\ -\xi^2 + \xi^2\eta + \eta^2 - \xi\eta^2 - \xi\eta + \eta + 1 \end{bmatrix};$$

when $1 - \xi - \eta \neq 0$. These yield

$$r(y) = \frac{\xi^2 - \xi^2\eta - \eta^2 + \xi\eta^2 - \xi\eta + \xi + 1}{(1-\xi-\eta)\eta} y^2 + c_y; \text{ and}$$

$$s(x) = \frac{-\xi^2 + \xi^2\eta + \eta^2 - \xi\eta^2 - \xi\eta + \eta + 1}{(1-\xi-\eta)\,\xi}x^2 + c_x;$$

where c_x, c_y are constants of integration.

One sufficient condition that satisfies the non-negativity requirement on transfers is that $c_x, c_y = \infty$. As this is unpalatable, it could be assumed that $c_x, c_y = 0$, in keeping with the reasoning behind equation 27 that previously set the constant to zero. In general, non-negativity would then require determination of which regions in (ξ, η) space satisfy it. But this is unnecessary: setting $c_x, c_y = 0$, reduces the transfers to quadratic equations, allowing application of the previous results. The ρ, σ, ξ, η combinations presented in equation 45 are consistent with this more general formulation.

Therefore:

Theorem 24 *Given linear emissions functions, continuous transfer functions capable of forming FNE to Problem 21 must be quadratic.*

6.2.3 The linear-quadratic FNE in the broader functional space

This section does not constrain emission and transfer functions to be linear or quadratic, as above. Instead, it merely requires that they be members of $\mathcal{C}^1$. The principal question asked here is whether the FNE derived above remain so in this broader functional space.

Lemma 25 *Agent 1's optimal emission function against*

$$y(z) = -z; s(x) = \frac{1}{3}x^2; \text{ and } r(y) = \frac{1}{3}y^2;$$

in Problem 21 is $x(z) = -z$.

PROOF. Agent 1's problem is to

$$\max_x E\left[-x^2 - z^2 - r(y) + s(x)\right];$$

subject to the Lemma's conditions and the state equation 1. These restrictions reduce the problem to

$$\max_z E\left[-4z^2 + \frac{8}{3}z\varepsilon - \frac{2}{3}\varepsilon^2\right];$$

whose first order condition sets $z = \frac{1}{3}\varepsilon$. The result follows by state equation 1 and $y = -z$. $\square$

The same technique determines that agent 1's optimal emission function against $y(z) = 3z$, $s(x) = \frac{1}{3}x^2$ and $r(y) = \frac{1}{3}y^2$ is $x(z) = 3z$.

It is more difficult to confirm that the quadratic transfers are optimal when not constrained. The technique just used in the lemma cannot be modified so that $x(z)$ is fixed but $r(y)$ is not as, given fixed emissions, agents will not transfer. Previously, when functional forms were constrained, the second stage calculations allowed derivation of best responses $\xi(\rho, \sigma)$ and $\eta(\rho, \sigma)$. These, substituted into the first stage, produced a problem in ρ and σ. This approach was possible as the assumption of quadratic transfer functions allowed these functions to be identified by a single parameter, something that cannot be done in general. As a result, only a conjecture is presented here:

Conjecture 26 *When admissible strategy functions in Problem 21 are members of $\mathcal{C}^1$, strategies supporting a FNE depend on the distribution of the stochastic variable, ε. Thus, linear emissions and quadratic transfers do not generally support a FNE.*

This conjecture is supported by initial calculations in which the agents' problems are reduced to optimal control problems of the standard forms in which the co-state equations depend on the distribution of ε.

7 Discussion

This paper has presented a number of results. First, a non-existence result was presented. Then, in a model with an FNE, the support of the stochastic variable, ε, was found only to trivially refine the set of functions supporting an FNE. This result contrasts with that in Klemperer and Meyer (1989).

In the model with transfers, equilibria have only been found when functions were restricted to be linear (for emissions) and quadratic (for transfers); they too are independent of the distribution of the stochastic variable. It is conjectured that this will not be the case when functions are less constrained.

The constraints on the functions in the game with transfers make the resulting FNE difficult to compare to those in the game without transfers. The comparison presented in Table 1 suggests that, while transfers may allow Pareto improvements even when they net out in equilibrium, they do not necessarily do so. It is unclear whether broadening the function spaces from which emission and transfer functions can be selected will Pareto improve or worsen the resulting FNE: if the game is regarded principally as a control problem then the broader control space can only be of benefit; if, though, it is regarded as a cooperation problem, then the extra freedom may give agents another means of engaging in strategic behaviour.

This paper also leaves open some further questions. Are there non-linear FNE in the quadratic model without transfers? Is it possible to make general statements about when additional control variables are Pareto improving? On the surface, this question resembles somewhat that of when adding an additional instrument in an incomplete market is Pareto improving.

Finally, the present agents are defined in a symmetric fashion, differing only by $\pm\delta$. To the extent that FNE are of interest, they are likely to be of interest when players may differ in other ways as well.

A Second order conditions in the transfer game

Partial differentiation of equations 28 and 29 with respect to ρ and σ produces

$$\begin{aligned} &\begin{bmatrix} 2\rho\eta + (\sigma-1)\,\xi & (\sigma-1)\,(\eta-1) & 0 & 0 \\ 0 & 0 & 2\rho\eta + (\sigma-1)\,\xi & (\sigma-1)\,(\eta-1) \\ (\sigma-1)\,(\xi-1) & 2\sigma\xi + (\rho-1)\,\eta & 0 & 0 \\ 0 & 0 & (\sigma-1)\,(\xi-1) & 2\sigma\xi + (\rho-1)\,\eta \end{bmatrix} \\ &\quad\times \begin{bmatrix} \eta_\rho & \xi_\rho & \eta_\sigma & \xi_\sigma \end{bmatrix}' \\ &= \begin{bmatrix} -\eta^2 & \xi\,(1-\eta) & \eta\,(1-\xi) & -\xi^2 \end{bmatrix}'; \end{aligned}$$

so that, when the 4×4 matrix is invertible,

$$\begin{aligned} &\Delta \begin{bmatrix} \eta_\rho & \xi_\rho & \eta_\sigma & \xi_\sigma \end{bmatrix}' \\ &= - \begin{bmatrix} \eta\,[(\sigma+1)\,\xi\eta + (\rho-1)\,\eta^2 + (\sigma-1)\,(\xi+\eta-1)] \\ (\xi-1)\,[(\rho+1)\,\eta^2 + (\sigma-1)\,\xi\eta] \\ (\eta-1)\,[(\sigma+1)\,\xi^2 + (\rho-1)\,\xi\eta] \\ \xi\,[(\rho+1)\,\xi\eta + (\sigma-1)\,\xi^2 + (\rho-1)\,(\xi+\eta-1)] \end{bmatrix}; \end{aligned} \tag{A.1}$$

where

$$\Delta \equiv 2\left[(\rho\eta + \sigma\xi)^2 - \rho\eta^2 - \sigma\xi^2\right] + (\rho\sigma - \rho - \sigma + 1)(\xi + \eta - 1)\,.$$

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